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Invariant variational problems on linear frame bundles

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Abstract
The Hamiltonian structure of variational problems defined by the natural basis $\mathcal{L}^i_{jk}$ of diff M-invariant Lagrangians on the 1-jet bundle of linear frames of a m-dimensional manifold M is described.

Diffeomorphism invariance on $J^1(FM)$ and its infinitesimal counterpart, i.e., invariance under the natural representation of vector fields of M, are analysed. The Lagrangians $\mathcal{L}^i_{jk}$ are proved to be the basic tools required to factor diff $M \times G$-invariance into diff M-invariance and G-invariance. The densities $\Omega^i_{jk} = \mathcal{L}^i_{jk}\theta^1 \wedge \cdots \wedge \theta^m$, where the θ^i are the components of the canonical form, are shown to define two types of variational problem according to whether $i \notin \{j, k\}$ or $i \in \{j, k\}$. The field equations for their extremals are deduced. These equations are examples of underdetermined non-linear systems of partial differential equations. Extremals defining a Lie algebra structure are characterized.

The functions in the real linear space spanned by $\mathcal{L}^i_{jk}$ are the only Lagrangians on FM admitting a Hamiltonian formalism of order zero. Infinitesimal symmetries and Noether invariants of the densities Ω^i_{jk} are studied in detail. In particular, it is proved that the Noether invariant of every vertical symmetry vanishes. Hence only the horizontal symmetries appear in the Hamiltonian structure. The equations of the Jacobi fields along an extremal are explicitly obtained.

The pre-symplectic structure attached to Ω^i_{jk} is defined to be an alternate bilinear map $(\omega_2)_s$ on the space of Jacobi fields along an extremal s with values in closed $(m-1)$-forms on M and its kernel is related to vertical infinitesimal symmetries of the Lagrangian. For $m = 3, 4$, the equations of the extremals are integrated explicitly; we thus obtain normal forms, which, when transformed by an arbitrary diffeomorphism, yield the general solution to the field equations.

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1. Introduction

Let $\pi\colon FM \to M$ be the bundle of linear frames of a m-dimensional manifold M. We define $\frac{1}{2}m^2(m-1)$ Lagrangians $\mathcal{L}^i_{jk}$ as follows: $\mathcal{L}^i_{jk}(j^1_x s) = \omega^i([X_j, X_k]_x)$, where $s = (X_1, \ldots, X_m)$ is a linear frame, i.e., a local section of π, and $(\omega^1, \ldots, \omega^m)$ is its dual coframe; for the details, see section 2.2 below. The purpose of the present paper is to describe the Hamiltonian structure of the variational problems defined by the densities $\Omega^i_{jk} = \mathcal{L}^i_{jk}\theta^1 \wedge \cdots \wedge \theta^m$, where the θ^i are the components of the canonical form on FM. The fundamental property of such Lagrangians is that they generate, under composition with differentiable functions and the total derivative, the ring of diffeomorphism-invariant Lagrangians on the jet bundles of linear frames. Diffeomorphism invariance is of interest in itself and plays an important role not only in classical general relativity, but also in supersymmetry and gauge theories [1, 5, 20, 27, 34] as it allows one to formulate the principle of general covariance for every specific situation.

The formulation of diff M-invariant variational principles on linear frame bundles is well known in several approaches to gravitation, such as tetrad or vierbein formalism [23, 36], Einstein–Cartan theory [3] and metric–affine theories (see [9, 12, 15, 17, 33]), and general relativity as a gauge theory [9, 10, 16, 19, 26]. This formulation is nothing but a translation to principal-bundle language of the classical 'anholonomic coordinates' in Cartan's moving frame theory; e.g., see [29, II, sections 9, 12]. The bundle of orthonormal frames is also used, especially in the 1 + 3 approach (e.g., see [35]), but the former formulation seems to be very suitable for dealing with a space-time with no preferred geometric decomposition. In any case, such an approach has the advantage of separating diffeomorphism invariance—a purely geometric condition—from the invariance under a given specific subgroup $G \subseteq GL(m; \mathbb{R})$. For a sound analysis showing the distinguished role of the bundle of linear frames in classical field theory, we refer the reader to [30].

From the structural point of view, the densities Ω^i_{jk} above present the most elementary diffeomorphism-invariant variational problems. Hence, although they are too simple to be of immediate application in field theory, precisely due to their simple properties, they provide interesting geometric models. In fact, each of the Lagrangians proposed as relativistic models on the bundle of linear frames can be written as a function of the basic Lagrangians $\mathcal{L}^i_{jk}$, as a result of which the $\mathcal{L}^i_{jk}$ are used as 'cornerstones' of the theory (see [30, 33]); also see section 2.3 for a discussion of the role that such Lagrangians play in imposing diff $M \times G$-invariance. We remark that the Lagrangians $\mathcal{L}^i_{jk}$ themselves cannot present any G-symmetry, as they generate the invariance under diff $M \times \{I\}$, I being the identity matrix.

The outline of the paper is as follows. In sections 2.1, 2.2, we define diff M-invariance on the 1-jet bundle of the linear frame bundle and its infinitesimal counterpart; i.e., invariance under the natural representation of vector fields of M into FM. Although the two definitions are essentially equivalent, they are not exactly the same due to some global topological obstructions on M, although they are essentially equivalent. We thus use the infinitesimal definition of invariance, as it allows us to employ tools such as vector distributions, involutiveness and the Frobenius theorem. In section 2.3 we show that the functions $\mathcal{L}^i_{jk}$ are the basic objects required to factor diff $M \times G$-invariance into diff M-invariance and G-invariance. This reveals the important role of the $\mathcal{L}^i_{jk}$ in formulating several relativistic theories based on linear frames (cf [12, 17, 33]). In section 2.4 we first prove that the densities Ω^i_{jk} define two types of variational problem according to whether $i \notin \{j, k\}$ or $i \in \{j, k\}$. If $\dim M = 2$, the density Ω^1_{12} is variationally trivial; hence, we assume $\dim M \geqslant 3$. Second, we obtain the field equations for the extremals of the action functional of Ω^i_{jk}. The number of equations is much lower than expected. In fact, as the standard fibre of FM is $GL(m; \mathbb{R})$, the number of Euler–

Lagrange equations is m^2 for such problems, but if $i \notin \{j, k\}$ (or $i \in \{j, k\}$) only $3(m-2)$ (or $3(m-1)$) of them are independent. Hence these equations are examples of underdetermined systems of PDEs (cf [2]), in contrast to overdetermined systems, which play a well-known role in classical field theory (cf [4]). By using these results we obtain two simple consequences: (1) integrable linear frames (i.e., with $[X_j, X_k] = 0$) are the common extremals of all Ω^i_{jk}; and (2) the characterization of extremals defining a Lie algebra structure; i.e., the extremals such that $[X_j, X_k] = c^i_{jk} X_i$; see [31] and proposition 2.8 below.

In section 3.1 we prove that the only diff M-invariant Lagrangians whose Poincaré–Cartan form projects onto FM (i.e., admitting a Hamiltonian formalism of order zero) are those of the vector space generated by $\mathcal{L}^i_{jk}$. This means that $\mathbb{R}$-linear combinations of $\mathcal{L}^i_{jk}$ are the only invariant Lagrangians having Euler–Lagrange equations of first order, thus providing a geometric meaning for this basis. In section 3.2 we determine the infinitesimal symmetries of Ω^i_{jk}. We first characterize the π-projectable symmetries that are common to all Ω^i_{jk} as being the natural lifts to FM of vector fields on M, and we determine the Noether invariants of such symmetries (theorem 3.2 and propositions 3.5, 3.6). It is a remarkable fact—stated in theorem 3.7—that the Noether invariant of every π-vertical symmetry vanishes. Hence only the 'horizontal' symmetries appear in the Hamiltonian structure. The equations of the Jacobi fields along an extremal are explicitly obtained in theorems 3.8, 3.9. Jacobi fields are thought of as being the tangent space for the 'manifold' of solutions at a given extremal. In section 3.3.2, we deduce conditions for a π-vertical vector field along an extremal s to be the vertical component of a horizontal symmetry, which, in addition, obliges s to admit a Lie algebra structure.

In section 3.4 we study the pre-symplectic structure attached to Ω^i_{jk}. This is defined to be an alternate bilinear map $(\omega_2)_s$ on the space of Jacobi fields along an extremal s. We prefer to consider $(\omega_2)_s$ as being a 2-form taking values in the space $Z^{m-1}(M)$ of closed $(m-1)$-forms on the ground manifold M, rather than a scalar form defined on a fixed compact $(m-1)$-dimensional domain, as in this way we can work independently of the domain of integration. In any case, the properties of the scalar pre-symplectic form can be recovered by simply integrating $(\omega_2)_s$ on a compact domain. The kernel of $(\omega_2)_s$ is then analysed. In proposition 3.14 we prove that if a Jacobi field X defined along s is an infinitesimal symmetry, then $i_X(\omega_2)_s = 0$. The converse is true if the linear frame s is integrable, while the outcome remains open for the non-integrable case. Finally, in section 4, the equations of the extremals of Ω^i_{jk} are integrated explicitly for $\dim M = m = 3, 4$, thus leading one to obtain 'normal forms'; i.e., transforming these normal forms by an arbitrary diffeomorphism, the general solution to the field equations is reached. Noether invariants defined by horizontal symmetries are also calculated in such dimensions.

2. Invariant Lagrangians on FM

2.1. diff *M-invariance and $\mathfrak{X}(M)$-invariance*

A Lagrangian density Ω_m defined on the 1-jet extension $J^1(FM)$ of the linear frame bundle $\pi\colon FM \to M$ of an m-dimensional manifold M is said to be diff M-invariant (or $\mathfrak{X}(M)$-invariant) if $J^1(\tilde{\phi})^*\Omega_m = \Omega_m$, $\forall \phi \in \operatorname{diff} M$ (or $L_{\tilde{X}^{(1)}}\Omega_m = 0$, $\forall X \in \mathfrak{X}(M)$), where $\tilde{\phi}$ (or $\tilde{X} \in \mathfrak{X}(FM)$) is the natural lift of ϕ (or X) to FM (see [13, VI, sections 1, 2]), and $\tilde{X}^{(1)}$ denotes the 1-jet prolongation of X; e.g., see [6, 8, 24, 28]. If $\theta = (\theta^1, \dots, \theta^m)$ is the canonical 1-form (see [13, III, section 2, p 118]), then we can write $\Omega_m = \mathcal{L}\theta^1 \wedge \cdots \wedge \theta^m$, where $\mathcal{L} \in C^\infty(J^1(FM))$ is called the canonical Lagrangian associated with Ω_m. A density Ω_m is diff M-invariant (or $\mathfrak{X}(M)$-invariant) if and only if $\mathcal{L} \circ J^1(\tilde{\phi}) = \mathcal{L}$, $\forall \phi \in \operatorname{diff} M$

(or $\tilde{X}^{(1)}\mathcal{L} = 0$, $\forall X \in \mathfrak{X}(M)$), as θ is both diff M-invariant and $\mathfrak{X}(M)$-invariant. Hence the problem of determining invariant Lagrangian densities is reduced to that of determining invariant Lagrangian functions.

Throughout the paper, italic indices run from 1 to m. Each coordinate system (x^i) on an open domain $U \subseteq M$ induces a coordinate system (x^i, x^i_j) on $\pi^{-1}(U)$, setting $u = ((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^m)_x) \cdot (x^i_j(u))$, $x = \pi(u)$, and a coordinate system $(x^i, x^i_j, x^i_{j,k})$ on J^1U. From the local expression

$$\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + x^h_j \frac{\partial u^i}{\partial x^h} \frac{\partial}{\partial x^i_j} + \left(x^h_j \frac{\partial^2 u^i}{\partial x^h \partial x^k} + \left(x^h_{j,k} \frac{\partial u^i}{\partial x^h} - x^i_{j,h} \frac{\partial u^h}{\partial x^k} \right) \right) \frac{\partial}{\partial x^i_{j,k}} \tag{1}$$

we conclude that a Lagrangian $\mathcal{L} \in C^\infty(J^1(FM))$ is $\mathfrak{X}(M)$-invariant if and only if it satisfies the following conditions:

$$0 = \frac{\partial \mathcal{L}}{\partial x^i} \tag{2}$$

$$0 = x^h_j \frac{\partial \mathcal{L}}{\partial x^i_j} + x^h_{j,k} \frac{\partial \mathcal{L}}{\partial x^i_{j,k}} - x^k_{j,i} \frac{\partial \mathcal{L}}{\partial x^k_{j,h}} \tag{3}$$

$$0 = x^h_j \frac{\partial \mathcal{L}}{\partial x^i_{j,k}} + x^k_j \frac{\partial \mathcal{L}}{\partial x^i_{j,h}}. \tag{4}$$

We denote by $\mathcal{I}_{\mathrm{diff}\,M}$ (or $\mathcal{I}_{\mathfrak{X}(M)}$) the algebra of diff M-invariant (or $\mathfrak{X}(M)$-invariant) Lagrangian functions on $J^1(FM)$. Obviously $\mathcal{I}_{\mathrm{diff}\,M} \subseteq \mathcal{I}_{\mathfrak{X}(M)}$, and $\mathcal{I}_{\mathrm{diff}\,M} = \mathcal{I}_{\mathfrak{X}(M)}$ except when M is orientable and admits an orientation-reversing diffeomorphism onto itself, in which case we have $\mathcal{I}_{\mathfrak{X}(M)} = \mathcal{I}_{\mathrm{diff}\,M} \times \mathcal{I}_{\mathrm{diff}\,M}$, $\mathcal{I}_{\mathrm{diff}\,M}$ being the diagonal of $\mathcal{I}_{\mathfrak{X}(M)}$; see [7] for the details.

Proposition 2.1. *If Ω_m is a* diff *M-invariant Lagrangian density on $J^1(FM)$, $s\colon U \to FM$ is an extremal of Ω_m and $\phi \in$* diff *U, then the section $\tilde{\phi} \circ s \circ \phi^{-1}$ is another extremal. In other words,* diff *M acts on the set of extremals of a* diff *M-invariant density.*

Proof. Let E^j_i be the $m \times m$ matrix $(E^j_i)^h_k = \delta^h_j \delta^i_k$, and let $\varepsilon^j_i \in V^*(FM)$ be the dual basis of the fundamental vector fields E^{j*}_i associated with E^j_i [13, I, section 4]; i.e., $\varepsilon^j_i(E^{l*}_k) = \delta^i_k \delta^l_j$. If M is oriented by a volume form v, then we have $\Omega_m = Lv$, $L \in C^\infty(J^1(FM))$. Let $(U; x^i)$ be coordinates such that $v = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ and let $\mathcal{E}(\Omega_m)\colon J^2(FM) \to V^*(FM) \otimes \wedge^m T^*M$, $\mathcal{E}(\Omega_m) \circ j^2 s = \mathrm{d}x^i_j \otimes (j^1 s)^* \mathcal{E}^j_i(L)$, be the Euler–Lagrange morphism

$$\mathcal{E}^j_i(L) = (-1)^h \,\mathrm{d}\left(\frac{\partial L}{\partial x^j_{i,h}} \right) \wedge v_h + \frac{\partial L}{\partial x^j_i} v \qquad v_h = \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^h} \wedge \cdots \wedge \mathrm{d}x^m. \tag{5}$$

By using the formulae $\theta^i = x_j{}^i\,\mathrm{d}x^j$, where $(x_j{}^i) = (x^i_j)^{-1}$, and $\mathrm{d}x^i_j = x^i_k \varepsilon^k_j$, we conclude that if $\mathcal{L} \in C^\infty(J^1(FM))$ is the Lagrangian associated with Ω_m, then there exist globally defined functions $\mathbb{E}^j_i(\mathcal{L}) \in C^\infty(J^2(FM))$, $\mathbb{E}^j_i(\mathcal{L}) = \det(x^a_b)(\mathcal{E}^j_k(L) x^k_i)$, such that $\mathcal{E}(\Omega_m)(j^2_x s) = \mathbb{E}^j_i(\mathcal{L})(j^2_x s)(\varepsilon^i_j)_{s(x)} \otimes (\theta^1 \wedge \cdots \wedge \theta^m)_{s(x)}$. Therefore, the Euler–Lagrange equations for Ω_m can globally be written as $\mathbb{E}^j_i(\mathcal{L}) \circ j^2 s = 0$. The result now follows from the functoriality of such functions; namely, $\mathbb{E}^j_i(\mathcal{L} \circ J^1(\tilde{\phi})) = \mathbb{E}^j_i(\mathcal{L}) \circ J^2(\tilde{\phi})$, $\forall \phi \in \mathrm{diff}\, M$ (e.g., see [14, XI, section 49] or [18]). In fact, as $\mathcal{L}$ is invariant, from the previous formula we obtain $\mathbb{E}^j_i(\mathcal{L}) \circ j^2 s = \mathbb{E}^j_i(\mathcal{L} \circ J^1(\tilde{\phi})) \circ j^2 s = \mathbb{E}^j_i(\mathcal{L}) \circ J^2(\tilde{\phi}) \circ j^2 s = \mathbb{E}^j_i(\mathcal{L}) \circ j^2(\tilde{\phi} \circ s \circ \phi^{-1})$. □

2.2. A basis for $\mathcal{I}_{\mathfrak{X}(M)}$

Let $\mathcal{L}^i_{jk}\colon J^1(FM) \to \mathbb{R}$, $j < k$, be the Lagrangian $\mathcal{L}^i_{jk}(j^1_x s) = \omega^i([X_j, X_k])(x)$, where $s = (X_1, \ldots, X_m)$ and $(\omega^1, \ldots, \omega^m)$ denotes the dual coframe. We remark that the definition makes sense as the value $\omega^i([X_j, X_k])(x)$ only depends on $j^1_x s$. Moreover, from the very definition we have $[X_j, X_k]_x = \mathcal{L}^i_{jk}(j^1_x s)(X_i)_x$. The local expression for $\mathcal{L}^i_{jk}$ is

$$\mathcal{L}^i_{jk} = (x^h_j x^l_{k,h} - x^h_k x^l_{j,h})x_l{}^i. \tag{6}$$

We claim that $\mathcal{L}^i_{jk}$ is diff M-invariant. In fact, for every $\phi \in$ diff M we have $J^1(\tilde{\phi})(j^1_x s) = j^1_{\phi(x)}(\tilde{\phi} \circ s \circ \phi^{-1})$, where $\tilde{\phi} \circ s \circ \phi^{-1} = (\phi \cdot X_1, \ldots, \phi \cdot X_m)$. Hence

$$[\phi \cdot X_j, \phi \cdot X_k]_{\phi(x)} = (\mathcal{L}^i_{jk} \circ J^1(\tilde{\phi}))(j^1_x s)(\phi \cdot X_i)_{\phi(x)}$$

and, taking into account that $\phi \cdot [X, Y] = [\phi \cdot X, \phi \cdot Y]$, we have

$$\phi_*([X_j, X_k]_x) = (\mathcal{L}^i_{jk} \circ J^1(\tilde{\phi}))(j^1_x s)\phi_*(X_i)_x = \phi_*((\mathcal{L}^i_{jk} \circ J^1(\tilde{\phi}))(j^1_x s)(X_i)_x)$$

which implies $(\mathcal{L}^i_{jk} \circ J^1(\tilde{\phi}))(j^1_x s) = \mathcal{L}^i_{jk}(j^1_x s)$, as ϕ_* is injective.

The Lagrangians $\mathcal{L}^i_{jk}$ are functionally independent and every $\mathcal{L} \in \mathcal{I}_{\mathfrak{X}(M)}$ can be written locally as a differentiable function of this system (see [7]).

2.3. G-invariance and teleparallelism theory

Let (v_i) be a basis of $\mathbb{R}^m$ with dual basis (v^i). The Lagrangians $\mathcal{L}^i_{jk}$ induce a natural map in the space of torsions; namely,

$$p\colon J^1(FM) \to \bigwedge^2 V^* \otimes V$$
$$p(j^1_x s) = \mathcal{L}^i_{jk}(j^1_x s)v^j \wedge v^k \otimes v_i$$

where $V = \mathbb{R}^m$. Note that $\mathcal{L}^i_{jk}(j^1_x s)$ are none other than the components of the opposite to the torsion tensor of the teleparallelism connection of the given linear frame (cf [33, section 2]); i.e., the connection parallelizing the vector fields $X_1, \ldots, X_m$; i.e.,

$$\nabla_{\partial/\partial x^h} X_j = 0. \tag{7}$$

The full linear group $GL(m; \mathbb{R})$ acts on $J^1(FM)$ by setting $j^1_x s \cdot A = j^1_x(R_A \circ s)$, where R_A denotes the right translation by the matrix $A \in GL(m; \mathbb{R})$, and it also acts on the space of torsions by the natural tensorial representation; i.e., $(t \cdot A)(x, y) = A^{-1}(t(A(x), A(y)))$ for every $t \in \wedge^2 V^* \otimes V$. Then, it is straightforward to prove that p is equivariant: precisely, $p(j^1_x s \cdot A) = p(j^1_x s) \cdot A$.

Let $G \subseteq GL(m; \mathbb{R})$ be a Lie subgroup. Several theories of gravitation, such as metric-teleparallel models (e.g., see [9,10,15–17,23]), are based on diff $M \times G$-invariant Lagrangians on $J^1(FM)$ for distinct choices of the group G; in particular, for $G = GL^+(m; \mathbb{R})$, $SL(m; \mathbb{R})$ and $O(k, m-k)$. As $(\mathcal{L}^i_{jk})$ is a basis for diff M-invariant Lagrangians, every diff M-invariant Lagrangian $\mathcal{L}$ can be written as $\mathcal{L} = F(\mathcal{L}^1_{12}, \ldots, \mathcal{L}^i_{jk}, \ldots, \mathcal{L}^m_{m-1,m})$ for a differentiable function F on $\wedge^2 V^* \otimes V$. Since p is surjective, $\mathcal{L}$ is G-invariant if and only if F is. Hence the problem of determining diff $M \times G$-invariant Lagrangians reduces to that of determining the invariant functions on the space of torsions under the action of the group G, which, essentially, is a question of algebraic nature because G-invariant functions admit an algebraic basis. For example, as the group $GL^+(m; \mathbb{R})$ is connected, a function $F \in C^\infty(\wedge^2 V^* \otimes V)$ is invariant if and only if it is infinitesimally invariant. If (u^i_{jk}) denote the coordinates in the basis $v^j \wedge v^k \otimes v_i$, then it is readily checked that invariance under the infinitesimal linear representation of $GL^+(m;\mathbb{R})$ is given by the following system of m^2 PDEs:

$$u^r_{ht}\frac{\partial F}{\partial u^r_{jt}} + u^r_{sh}\frac{\partial F}{\partial u^r_{sj}} - u^j_{st}\frac{\partial F}{\partial u^h_{st}} = 0. \qquad (8)$$

As these equations are independent and constitute an involutive system, by simply applying the Frobenius theorem we conclude that the number of invariant functions is, in this case, equal to $\frac{1}{2}m^2(m-1) - m^2 = \frac{1}{2}m^2(m-3)$ for $m \geqslant 4$. Specific examples can be found in [30, section 5], [31, sections 1.1–1.3], [33, sections 2 and 3] for different choices of the function F. The other subgroups can be dealt with similarly.

The previous procedure can be inverted: first we can require $GL(m;\mathbb{R})$-invariance and then we require diff M-invariance. We know that $GL(m;\mathbb{R})$-invariant functions on $J^1(FM)$ are the functions on the quotient bundle $J^1(FM)/GL(m;\mathbb{R})$, which can be identified with the bundle of linear connections $C(M)$ of M. In fact, by assigning its associated connection to each linear frame, say $j^1_x s \mapsto \nabla_x$, we obtain a projection $p\colon J^1(FM) \to C(M)$. If $X_j = f^i_j \partial/\partial x^i$, then by imposing the equations (7) we obtain the following relations for the local symbols of the connection: $\Gamma^k_{hi} f^i_j = -\partial f^k_j/\partial x^h$. Denoting by (x^i, A^j_{kl}) the standard coordinates on the bundle of connections, we conclude that the equations for p are $p^*(A^k_{hr}) = -x_r{}^j x^k_{j,h}$. From these expressions it is readily checked that the fibres of p coincide with the orbits of $GL(m;\mathbb{R})$. This means that two jets $j^1_x s$, $j^1_x s'$ are $GL(m;\mathbb{R})$-equivalent if and only if they define the same connection at $x \in M$. Moreover, we have a natural map $\tau\colon C(M) \to \wedge^2 T^*M \otimes TM$ which associates a torsion tensor with each linear connection. If (x^i, t^j_{kl}) are the natural coordinates on $\wedge^2 T^*M \otimes TM$, then the equations of the map τ are the following: $\tau^*(t^i_{jk}) = A^i_{jk} - A^i_{kj}$, $j < k$. We claim that the following diagram commutes:

$$\begin{array}{ccc} J^1(FM) & \xrightarrow{q} & C(M) \\ {\scriptstyle (\pi^1_0, p)} \downarrow & & \downarrow {\scriptstyle \tau} \\ FM \times \bigwedge^2 V^* \otimes V & \xrightarrow{\varrho} & \bigwedge^2 T^*M \otimes TM \end{array}$$

where ϱ maps the pair $(u, \lambda^i_{jk} v^j \wedge v^k \otimes v_i)$ onto the tensor whose coordinates are the scalars λ^i_{jk} in the frame u. Then, commutativity follows from the standard construction of the associated bundle (e.g., see [13, I, section 5]). In addition, the projections q and τ are equivariant with respect to the natural representations of diffeomorphisms on $J^1(FM)$, $C(M)$, and $\wedge^2 T^*M \otimes TM$, respectively. Hence we conclude that the problem of computing $GL(m;\mathbb{R})$-invariants on the vector space $\wedge^2 V^* \otimes V$ is equivalent to computing diff M-invariants on the bundle of torsions $\wedge^2 T^*M \otimes TM$. In fact, as a computation shows, a function $F \in C^\infty(\wedge^2 T^*M \otimes TM)$ is infinitesimally diff M-invariant if and only if the following system holds:

$$\frac{\partial F}{\partial x^i} = 0$$

$$t^r_{ht}\frac{\partial F}{\partial t^r_{jt}} + t^r_{sh}\frac{\partial F}{\partial t^r_{sj}} - t^j_{st}\frac{\partial F}{\partial t^h_{st}} = 0.$$

Note that this system is exactly equivalent to the system (8).

Finally, we also remark that the functions $\mathcal{L}^i_{jk}$ themselves cannot be invariant under any proper subgroup because they constitute a basis for the invariance under diff $M \times \{I\}$, I being the identity matrix.

2.4. Extremals of $\mathcal{L}^i_{jk}$

Proposition 2.2 (cf [25]). *Let $\Omega^i_{jk} = \mathcal{L}^i_{jk}\theta^1 \wedge \cdots \wedge \theta^m$, where $\mathcal{L}^i_{jk}$ are the Lagrangians of the formula (6). The variational problems defined by Ω^i_{jk}, $i \notin \{j,k\}$, are structurally equal to each other and also those defined by Ω^i_{jk}, $i \in \{j,k\}$, are structurally the same. Therefore, the densities Ω^i_{jk} define two types of variational problem according to whether $i \notin \{j,k\}$ or $i \in \{j,k\}$. If* $\dim M = 2$*, the density $\Omega^1_{12} = \mathcal{L}^1_{12}\theta^1 \wedge \theta^2$ is variationally trivial. Hence, in what follows we assume* $\dim M \geqslant 3$.

Proof. Let us consider the case $i \notin \{j,k\}$. The other case is dealt with similarly. Let I be the set $\{(i,j,k) | j < k, i \notin \{j,k\}\}$. Given two systems of indices $(i,j,k), (a,b,c) \in I$, there exists $\sigma \in \mathrm{perm}\{1,\ldots,m\}$ such that $\sigma(i) = a$, $\sigma(j) = b$, $\sigma(k) = c$. Let $\Psi_\sigma \in \mathrm{diff}\, FM$ be defined by $\Psi_\sigma(X_1,\ldots,X_m) = (X_{\sigma(1)},\ldots,X_{\sigma(m)})$. Then, $\Psi_{\sigma^{-1}}$ transforms bijectively the extremals of Ω^i_{jk} onto those of Ω^a_{bc}: if s is an extremal of Ω^i_{jk} defined on an m-dimensional compact submanifold with boundary $N \subseteq M$, then $\Psi_{\sigma^{-1}} \circ s$ is an extremal of Ω^a_{bc}. In fact, if S_t is a one-parameter variation of $\Psi_{\sigma^{-1}} \circ s$, then $\Psi_\sigma \circ S_t$ is a one-parameter variation of s, and taking into account that $\Psi^*_\sigma\theta^i = x_{\sigma(j)}{}^i\, \mathrm{d}x^j$, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N (j^1(\Psi_\sigma \circ S_t))^*\Omega^i_{jk} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N (j^1 S_t)^* \circ (J^1\Psi_\sigma)^*\Omega^i_{jk} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N (j^1 S_t)^*((\mathcal{L}^i_{jk} \circ J^1\Psi_\sigma)\Psi^*_\sigma\theta^1 \wedge \cdots \wedge \Psi^*_\sigma\theta^m) \\
&= \varepsilon(\sigma)\, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N (j^1 S_t)^*\Omega^a_{bc}
\end{aligned}$$

where $\varepsilon(\sigma)$ denotes the sign of σ, thus finishing the proof. □

Theorem 2.3. *The section $s = (X_1,\ldots,X_m)$ of FM with dual coframe $(\omega^1,\ldots,\omega^m)$ is an extremal of Ω^1_{23} if and only if the following* $3(m-2)$ *equations hold:*

$$\begin{aligned}
&(a)\ \mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^4 \wedge \cdots \wedge \overset{(i)}{\omega^j} \wedge \cdots \wedge \omega^m = 0 \qquad 4 \leqslant i \leqslant m \qquad j = 2,3 \\
&(b)\ \omega^j \wedge \mathrm{d}(\omega^1 \wedge \omega^4 \wedge \cdots \wedge \omega^m) + \mathrm{d}\omega^1 \wedge \omega^j \wedge \omega^4 \wedge \cdots \wedge \omega^m = 0 \qquad j = 2,3 \\
&(c)\ \mathrm{d}\omega^j \wedge \omega^1 \wedge \omega^4 \wedge \cdots \wedge \omega^m = 0 \qquad j \neq 2,3
\end{aligned}$$

Proof. If $s\colon N \to FM$ is a section, then $(j^1 s)^*\Omega^1_{23} = \mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^4 \wedge \cdots \wedge \omega^m$. Let s be an extremal of Ω^1_{23}. Let E^j_i be as in the proof of proposition 2.1. Let $S_t(x) = s(x)\exp(t\varphi(x)E^j_i)$, $|t| < \varepsilon$, $\varphi \in C^\infty(M)$, be a one-parameter variation of s with $\sup\varphi \subset N \setminus \partial N$. We have

$$\exp(t\varphi(x)E^j_i) = \begin{cases} I + t\varphi(x)E^j_i & i \neq j \\ I + (\mathrm{e}^{t\varphi(x)} - 1)E^i_i & i = j. \end{cases}$$

The dual coframe of $S_t = (X^t_1,\ldots,X^t_m)$ is $(\omega^1_t,\ldots,\omega^m_t) = (\omega^1,\ldots,\omega^m)\cdot\exp(-t\varphi E^i_j)$, where $\omega^l_t = (\delta^l_k - t\varphi(E^i_j)^l_k)\omega^k = (\delta^l_k - t\varphi\delta^j_k\delta^l_i)\omega^k = \omega^l - t\varphi\delta^l_i\omega^j$ for $i \neq j$. Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N (j^1 S_t)^*\Omega^1_{23} &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_N \mathrm{d}\omega^1_t \wedge \omega^1_t \wedge \omega^4_t \wedge \cdots \wedge \omega^m_t \\
&= -\int_N [\mathrm{d}(\varphi\delta^1_i\omega^j) \wedge \omega^1 + \varphi\delta^1_i\, \mathrm{d}\omega^1 \wedge \omega^j] \wedge \omega^4 \wedge \cdots \wedge \omega^m \\
&\quad - \sum_{k=4}^m \delta^k_i \int_N \varphi\, \mathrm{d}\omega^1 \wedge \omega^1 \wedge \omega^4 \wedge \cdots \wedge \overset{(k)}{\omega^j} \wedge \cdots \wedge \omega^m.
\end{aligned}$$

Taking into account that $\sup\varphi \subset N\setminus\partial N$, from Stokes's theorem we have

$$\int_N \mathrm{d}(\varphi\delta_i^1\omega^j)\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m = \int_N \varphi\delta_i^1\omega^j\wedge\mathrm{d}(\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m).$$

Since s is an extremal, for every $\varphi\in C^\infty(M)$ we obtain

$$0=\int_N\varphi\Bigg[\delta_i^1\omega^j\wedge\mathrm{d}(\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m)+\delta_i^1\,\mathrm{d}\omega^1\wedge\omega^j\wedge\omega^4\wedge\cdots\wedge\omega^m + \sum_{k=4}^m\delta_i^k\,\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overset{k)}{\omega^j}\wedge\cdots\wedge\omega^m\Bigg].$$

By applying the fundamental lemma of the calculus of variations, for $i=1$, $j=2,3$ (or $j\neq 2,3$) we obtain (b) (or (c)). For $i=2,3$, or $4\leqslant i\leqslant m$, $j\neq 2,3$ the previous equation is trivial and for $4\leqslant i\leqslant m$, $j=2,3$ we obtain (a). Conversely, assume s satisfies (a)–(c). Let S be a one-parameter variation of s on N. Then, for $|t|<\varepsilon$ we have two linear frames $S_t(x)=S(t,x)$ and $s(x)$ at $x\in N$. Therefore, there exists a unique $A\colon(-\varepsilon,\varepsilon)\times N\to GL(m;\mathbb{R})$, $A=(a_j^i)$, such that $S(t,x)=s(x)\cdot A(t,x)$ with $A(0,x)=I$, $\forall x\in N$; $A(t,x)=I$, $\forall x\in M\setminus N$. Hence, $A(t,x)=I+t\partial A/\partial t(0,x)+t^2B(t,x)$, $B(t,x)$ being an $m\times m$ matrix. Let $\varphi_j^i(x)$ be the entries of the matrix $\partial A/\partial t(0,x)$. Then, the dual coframe of $(X_1^t,\ldots,X_m^t)=(X_1,\ldots,X_m)\cdot A$ is $(\omega_t^1,\omega_t^2,\ldots,\omega_t^m)$, $\omega_t^l={a_j}^l\omega^j$, where $({a_j}^l)=A^{-1}$. Accordingly, we have

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0}(j^1S_t)^*\Omega_{23}^1 &= \frac{\partial}{\partial t}\bigg|_{t=0}(\mathrm{d}\omega_t^1\wedge\omega_t^1\wedge\omega_t^4\wedge\cdots\wedge\omega_t^m)\\ &=\left(\mathrm{d}\left(\frac{\partial {a_j}^1}{\partial t}\bigg|_{t=0}\omega^j\right)\wedge\omega^1+\mathrm{d}\omega^1\wedge\frac{\partial {a_j}^1}{\partial t}\bigg|_{t=0}\omega^j\right)\wedge\omega^4\wedge\cdots\wedge\omega^m\\ &\quad+\sum_{k=4}^m\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overbrace{\frac{\partial {a_j}^k}{\partial t}\bigg|_{t=0}\omega^j}^{k)}\wedge\cdots\wedge\omega^m.\end{aligned}$$

From $a_k^h{a_j}^k=\delta_j^h$ we obtain $(\partial {a_j}^h/\partial t)(0,x)=-(\partial a_j^h/\partial t)(0,x)=-\varphi_j^h(x)$. Hence, the equation above yields

$$\begin{aligned}&-[\mathrm{d}(\varphi_j^1\omega^j)\wedge\omega^1+\varphi_j^1\,\mathrm{d}\omega^1\wedge\omega^j]\wedge\omega^4\wedge\cdots\wedge\omega^m\\ &\quad-\sum_{k=4}^m\varphi_j^k\,\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overset{k)}{\omega^j}\wedge\cdots\wedge\omega^m\\ &=-\mathrm{d}\varphi_j^1\wedge\underbrace{\omega^j\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m}_{=0,\ j\neq2,3}-\varphi_j^1\underbrace{\mathrm{d}\omega^j\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m}_{\overset{(c)}{=}0,\ j\neq2,3}\\ &\quad-\varphi_j^1\underbrace{\mathrm{d}\omega^1\wedge\omega^j\wedge\omega^4\wedge\cdots\wedge\omega^m}_{=0,\ j\neq2,3}\\ &\quad-\sum_{k=4}^m\varphi_j^k\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overset{k)}{\omega^j}\wedge\cdots\wedge\omega^m}_{=0,\ j\neq2,3}\\ &=-\sum_{j=2,3}\Big(\mathrm{d}(\varphi_j^1\omega^j\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m)\\ &\quad+\varphi_j^1\underbrace{\{\omega^j\wedge\mathrm{d}(\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m)+\mathrm{d}\omega^1\wedge\omega^j\wedge\omega^4\wedge\cdots\wedge\omega^m\}}_{\overset{(b)}{=}0}\end{aligned}$$

$$+\sum_{k=4}^{m}\varphi_j^k\underbrace{\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overset{k)}{\omega^j}\wedge\cdots\wedge\omega^m}_{\overset{(a)}{=}0}\Bigg)$$

$$=-\sum_{j=2,3}\mathrm{d}(\varphi_j^1\omega^j\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m).$$

Since $\sup\varphi_j^1\subset N\setminus\partial N$, from Stokes's theorem we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\int_N((j^1S_t)^*\Omega_{23}^1)=-\int_{\partial N}\varphi_j^1\omega^j\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\omega^m=0$$

which completes the proof. □

Corollary 2.4. *A section $s=(X_1,\ldots,X_m)$ of FM is an extremal of Ω_{23}^1 if and only if the following $3(m-2)$ equations hold:*

$$\omega^1([X_j,X_i])=0 \qquad 4\leqslant i\leqslant m,\ j=2,3$$

$$2\omega^1([X_j,X_1])+\sum_{l=4}^{m}\omega^l([X_j,X_l])=0 \qquad j=2,3$$

$$\omega^j([X_2,X_3])=0 \qquad j\neq 2,3.$$

Proof. Since $\mathrm{d}\omega^1\wedge\omega^1\wedge\omega^4\wedge\cdots\wedge\overset{i)}{\omega^2}\wedge\cdots\wedge\omega^m=-\omega^1([X_3,X_i])\omega^1\wedge\cdots\wedge\omega^m$, from (a) in theorem 2.3 we obtain the first equation above for $j=3$. In the same way, we obtain the rest of the equations. □

Let $(U;x^i)$ be a coordinate system on the domain of a linear frame $s=(X_1,\ldots,X_m)$, such that $X_j=f_j^i\partial/\partial x^i$, $f_j^i\in C^\infty(U)$. Then, s is an extremal of Ω_{23}^1 if and only if:

$$\begin{aligned}
&\left(f_j^k\frac{\partial f_i^h}{\partial x^k}-f_i^k\frac{\partial f_j^h}{\partial x^k}\right)f_h{}^1=0 && 4\leqslant i\leqslant m,\ j=2,3\\
&\left(f_1^k\frac{\partial f_j^h}{\partial x^k}-f_j^k\frac{\partial f_1^h}{\partial x^k}\right)f_h{}^1+\sum_{l=4}^{m}\left(f_l^k\frac{\partial f_j^h}{\partial x^k}-f_j^k\frac{\partial f_l^h}{\partial x^k}\right)f_h{}^l=0 && j=2,3\\
&\left(f_2^k\frac{\partial f_3^h}{\partial x^k}-f_3^k\frac{\partial f_2^h}{\partial x^k}\right)f_h{}^j=0 && j\neq 2,3.
\end{aligned}\qquad(9)$$

Similarly, we have

Theorem 2.5. *A section $s=(X_1,\ldots,X_m)$ of FM is an extremal of Ω_{12}^1 if and only if the following $3(m-1)$ equations hold:*

$$(a)\ \mathrm{d}\omega^1\wedge\omega^3\wedge\omega^4\wedge\cdots\wedge\overset{(i)}{\omega^j}\wedge\cdots\wedge\omega^m=0 \qquad 3\leqslant i\leqslant m,\ j=1,2$$

$$(b)\ \omega^j\wedge\mathrm{d}(\omega^3\wedge\omega^4\wedge\cdots\wedge\omega^m)=0 \qquad j=1,2$$

$$(c)\ \mathrm{d}\omega^j\wedge\omega^3\wedge\omega^4\wedge\cdots\wedge\omega^m=0 \qquad j\neq 2.$$

Corollary 2.6. *A section $s=(X_1,\ldots,X_m)$ of FM is an extremal of Ω_{12}^1 if and only if the following $3(m-1)$ equations hold:*

$$\omega^1([X_j,X_i])=0 \qquad 3\leqslant i\leqslant m,\ j=1,2$$

$$\sum_{l=3}^{m}\omega^l([X_j,X_l])=0 \qquad j=1,2$$

$$\omega^j([X_1,X_2])=0 \qquad j\neq 2.$$

Let $(U; x^i)$ be a coordinate system on the domain of a linear frame $s = (X_1, \dots, X_m)$, such that $X_j = f_j^i \partial/\partial x^i$, $f_j^i \in C^\infty(U)$. Then, s is an extremal of Ω_{12}^1 if and only if:

$$\begin{aligned} &\left(f_j^k \frac{\partial f_i^h}{\partial x^k} - f_i^k \frac{\partial f_j^h}{\partial x^k} \right) f_h{}^1 = 0 && 3 \leqslant i \leqslant m,\ j = 1, 2 \\ &\sum_{l=3}^m \left(f_l^k \frac{\partial f_j^h}{\partial x^k} - f_j^k \frac{\partial f_l^h}{\partial x^k} \right) f_h{}^l = 0 && j = 1, 2 \\ &\left(f_2^k \frac{\partial f_1^h}{\partial x^k} - f_1^k \frac{\partial f_2^h}{\partial x^k} \right) f_h{}^j = 0 && j \neq 2. \end{aligned} \tag{10}$$

By using the proof of proposition 2.2, from the corollaries 2.4, 2.6 we obtain the extremals of Ω_{jk}^i: for $i \notin \{j, k\}$, we have

$$\begin{aligned} &\omega^i([X_a, X_b]) = 0 && a \neq i, j, k,\ b = j, k \\ &\omega^i([X_b, X_i]) + \sum_{r \neq j,k} \omega^r([X_b, X_r]) = 0 && b = j, k \\ &\omega^b([X_j, X_k]) = 0 && b \neq j, k \end{aligned} \tag{11}$$

and for $i \in \{j, k\}$,

$$\begin{aligned} &\omega^i([X_a, X_b]) = 0 && a \neq i, j,\ b = i, j \\ &\sum_{r \neq i,j} \omega^r([X_b, X_r]) = 0 && b = i, j \\ &\omega^b([X_i, X_j]) = 0 && b \neq j. \end{aligned} \tag{12}$$

Proposition 2.7. *A frame $s = (X_1, \dots, X_m)\colon U \to FM$ is an extremal of all Lagrangian densities Ω_{jk}^i if and only if for every $x_0 \in U$ there exists a coordinate system $(U; x^i)$ on M such that $X_i|_U = \partial/\partial x^i|_U$.*

Proof. From the equations (11), (12) it follows that $(\partial/\partial x^1, \dots, \partial/\partial x^m)$ is a common extremal of the Lagrangian densities Ω_{jk}^i. For the converse it is enough to prove $[X_i, X_j] = 0$. Let us fix a pair of indices $i < j$, and let us set $\chi_{bc}^a = \mathcal{L}_{bc}^a \circ j^1 s = \omega^a([X_b, X_c])$. If s is an extremal of Ω_{ri}^r, $r < i$, then the third equation in (12) can be rewritten as $0 = \chi_{ri}^r = -\chi_{ir}^r$. Similarly, the second equation in (12) for Ω_{ij}^i yields

$$0 = \sum_{r \neq i,j} \chi_{ir}^r = \sum_{i<r,\, r \neq i,j} \chi_{ir}^r + \sum_{i>r,\, r \neq i,j} \chi_{ir}^r = \sum_{i<r,\, r \neq i,j} \chi_{ir}^r. \tag{13}$$

Therefore, for each index $j > i$, we obtain

$$\begin{array}{llllll} j = i+1 & & \chi_{i,i+2}^{i+2} & +\chi_{i,i+3}^{i+3} + \cdots + \chi_{i,m-1}^{m-1} + \chi_{i,m}^m & = 0 \\ j = i+2 & \chi_{i,i+1}^{i+1} & & +\chi_{i,i+3}^{i+3} + \cdots + \chi_{i,m-1}^{m-1} + \chi_{i,m}^m & = 0 \\ \vdots & & & \vdots & \\ j = m & \chi_{i,i+1}^{i+1} + & \chi_{i,i+2}^{i+2} & +\chi_{i,i+3}^{i+3} + \cdots + \chi_{i,m-1}^{m-1} & = 0 \end{array}$$

where, by subtracting the first equation from the second one, we deduce $\chi_{i,i+1}^{i+1} = \chi_{i,i+2}^{i+2}$. By repeating the same process successively we have $\chi_{i,i+1}^{i+1} = \chi_{i,i+2}^{i+2} = \chi_{i,i+3}^{i+3} = \cdots = \chi_{i,m}^m$. Hence equation (13) reduces to $(m - i - 1)\chi_{ir}^r = 0$, for all $r > i$. Thus

$$\chi_{ir}^r = 0 \qquad \text{for all } i < m - 1,\ r > i. \tag{14}$$

As s is an extremal of $\Omega_{m-2,m-1}^{m-3}$, from the second equation in (11) and (14) we conclude that

$$0 = \chi_{m-1,m-3}^{m-3} + \sum_{r \neq m-1,m-2} \chi_{m-1,r}^r = \chi_{m-1,m}^m$$

which completes the proof. □

The number of the Euler–Lagrange equations for the extremals of a Lagrangian on FM is m^2, a number much greater than that of the equations (11), (12) defining the extremals of the densities Ω^i_{jk}. We end this section by showing that these equations are really equivalent to the Euler–Lagrange equations of such densities. According to proposition 2.2 we only need to do this for Ω^1_{23}, Ω^1_{12}. We give the proof for Ω^1_{23}, the other case being similar.

Let $(J^1(FU); x^i, x^i_j, x^i_{j,k})$ be the system induced by $(U; x^i)$. Set $L^1_{23} = \mathcal{L}^1_{23}\det(x^a_b)$. First we prove that the Euler–Lagrange equations for the extremals of Ω^1_{23} can be written as $\Psi^k_j \circ j^1 s = 0$, where $\Psi^k_j \colon J^1(FU) \to \mathbb{R}$, are the following functions:

$$\Psi^k_j = (-\delta^k_3 x^i_{2,i} + \delta^k_2 x^i_{3,i})x^1_j + (\delta^k_3 x^i_2 - \delta^k_2 x^i_3)({x_q}^1 {x_j}^r + x^1_j {x_q}^r)x^q_{r,i} + (\delta^k_2 x^l_{3,j} - \delta^k_3 x^l_{2,j}){x_l}^1 - (x^h_2 x^l_{3,h} - x^h_3 x^l_{2,h})({x_j}^1 {x_l}^k + {x_l}^1 {x_j}^k). \tag{15}$$

In fact, the Euler–Lagrange equations of Ω^1_{23} are (see the formula (5))

$$(j^1 s)^* \mathcal{E}^k_j(L^1_{23}) = (j^1 s)^* \left((-1)^i \,\mathrm{d}\left(\frac{\partial L^1_{23}}{\partial x^j_{k,i}}\right) \wedge v_i + \frac{\partial L^1_{23}}{\partial x^j_k} v \right) = 0. \tag{16}$$

Taking into account the expression (6) of $\mathcal{L}^1_{23}$, we obtain

$$\begin{aligned}\mathcal{E}^k_j(L^1_{23}) &= (-1)^i \,\mathrm{d}((\delta^k_3 x^i_2 - \delta^k_2 x^i_3){x_j}^1 \det({x_b}^a)) \wedge v_i + \frac{\partial L^1_{23}}{\partial x^j_k} v \\ &= F^{i,kr}_{jq} \vartheta^q_r \wedge v_i + \Psi^k_j \det({x_b}^a) v \end{aligned} \tag{17}$$

for certain functions $F^{i,kr}_{jq} \in C^\infty(J^1(FU))$ where $\vartheta^q_r = \mathrm{d}x^q_r - x^q_{r,i}\,\mathrm{d}x^i$ are the standard contact forms on $J^1(FM)$ [7, section 1.3]. Hence $(j^1 s)^* \mathcal{E}^k_j(L^1_{23}) = (\Psi^k_j \circ j^1 s)\det({f_b}^a)v$, with $s = (X_1, \ldots, X_m)$, $X_j = f^i_j \partial/\partial x^i$. From (15) it is easily seen that

$$(\Psi^i_j)\cdot(x^i_j) = \begin{pmatrix} -2\mathcal{L}^1_{23} & 0 & 0 & 0 & \ldots & 0 \\ 2\mathcal{L}^1_{31} + \sum_{l=4}^m \mathcal{L}^l_{3l} & 0 & -\mathcal{L}^1_{34} & -\mathcal{L}^1_{35} & \ldots & -\mathcal{L}^1_{3m} \\ -(2\mathcal{L}^1_{21} + \sum_{l=4}^m \mathcal{L}^l_{2l}) & 0 & \mathcal{L}^1_{24} & \mathcal{L}^1_{25} & \ldots & \mathcal{L}^1_{2m} \\ -\mathcal{L}^4_{23} & 0 & -\mathcal{L}^1_{23} & 0 & \ldots & 0 \\ -\mathcal{L}^5_{23} & 0 & 0 & -\mathcal{L}^1_{23} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathcal{L}^m_{23} & 0 & 0 & 0 & \ldots & -\mathcal{L}^1_{23} \end{pmatrix}. \tag{18}$$

Pulling this equation back along $j^1 s$ and taking into account that $\mathcal{L}^a_{bc} \circ j^1 s = \omega^a([X_b, X_c])$ and that the matrix (f^i_j) is invertible from corollary 2.4, we have completed our exposition.

2.5. Extremals with a Lie algebra structure

Let $(X_1, \ldots, X_m)$ be a linear frame defining a Lie algebra structure; that is, $[X_i, X_j] = c^h_{ij} X_h$ for certain structure constants c^h_{ij}. These linear frames are of interest by virtue of the third theorem of Lie (see e.g., [11, II, theorem 7.5]), which states that every Lie algebra can be obtained in the previous form, and they also seem to be important in field theory [31, 33]. Let us determine the conditions on the structure constants for such a Lie algebra to be an extremal of Ω^1_{23} or Ω^1_{12}. In fact, from the formulae (11), (12) we have

Proposition 2.8. *A linear frame $(X_1, \ldots, X_m)$ admitting a Lie algebra structure is an extremal of Ω^i_{jk}, $i \notin \{j, k\}$, (or Ω^i_{ij}) if and only if the structure constants satisfy $c^i_{ab} = 0$, $a \neq i, j, k$, $b = j, k$; $c^i_{bi} + \sum_{r \neq j,k} c^r_{br} = 0$, $b = j, k$; $c^b_{jk} = 0$, $b \neq j, k$ (or $c^i_{ab} = 0$, $a \neq i, j$, $b = i, j$; $\sum_{r \neq i,j} c^r_{br} = 0$, $b = i, j$; $c^b_{ij} = 0$, $b \neq j$).*

If $m = \dim M = 3$, then we have:

(i) The linear frame (X_1, X_2, X_3) is an extremal for Ω^i_{jk}, $i \notin \{j, k\}$, if and only if the vector space spanned by X_j, X_k is an ideal of the Lie algebra $\langle X_1, X_2, X_3\rangle$.

(ii) The linear frame (X_1, X_2, X_3) is an extremal for Ω^i_{ij} if and only if either X_j is non-central, and then $\langle X_1, X_2, X_3\rangle$ is the direct sum of $\langle X_j\rangle$ and the non-Abelian two-dimensional Lie algebra, or X_j is central, and then $\langle X_1, X_2, X_3\rangle$ is the Lie algebra of strictly upper triangular matrices.

3. Hamiltonian structure

3.1. The Poincaré–Cartan form of Ω^i_{jk}

The Poincaré–Cartan form of a Lagrangian density Lv on a fibred manifold $p: P \to M$, with $\dim P = m + n$, is the m-form on J^1P given on a fibred coordinate system $(x^i, y^\alpha, y^\alpha_i)$, $1 \leqslant \alpha \leqslant n$, by (see [6, 8])

$$\Theta = (-1)^{i-1}\frac{\partial L}{\partial y^\alpha_i}\vartheta^\alpha \wedge v_i + Lv \tag{19}$$

where $\vartheta^\alpha = \mathrm{d}y^\alpha - y^\alpha_i\,\mathrm{d}x^i$ are the standard contact forms (see [7, section 1.3]). Taking into account (6) and (19), a simple calculation shows that the Poincaré–Cartan form of the Lagrangian density Ω^i_{jk} is

$$\Theta^i_{jk} = (-1)^{h-1}x_r{}^i \det(x_b{}^a)(x^h_j\,\mathrm{d}x^r_k - x^h_k\,\mathrm{d}x^r_j) \wedge v_h. \tag{20}$$

Hence the Poincaré–Cartan form Θ^i_{jk} of Ω^i_{jk} is projectable onto $J^0(FM) = FM$. Conversely,

Theorem 3.1. *The Poincaré–Cartan form Θ of a $\mathfrak{X}(M)$-invariant Lagrangian density $\Omega_m = \mathcal{L}\theta^1 \wedge \cdots \wedge \theta^m$ on $J^1(FM)$ is projectable onto $J^0(FM) = FM$ if and only if there exist $\lambda, \lambda^{jk}_i \in \mathbb{R}$ such that $\mathcal{L} = \lambda + \lambda^{jk}_i\mathcal{L}^i_{jk}$.*

Proof. From (19) it follows that the Poincaré–Cartan form Θ is projectable onto FM if and only if $\partial L/\partial x^i_j$ and $L - x^i_{j,k}\partial L/\partial x^i_{j,k}$ project onto FU, $U \subset M$ being the open subset where the coordinates (x^i) are defined and $L = \mathcal{L}\det(x_b{}^a)$. This means that L is an affine function; i.e., $L = f^{jk}_i x^i_{j,k} + f$, with f^{jk}_i, $f \in C^\infty(FU)$. Hence $\mathcal{L} = g^{jk}_i x^i_{j,k} + g$, with $g = f\det(x^a_b)$, $g^{jk}_i = f^{jk}_i\det(x^a_b)$. As $\mathcal{L}$ is invariant, from (2) it follows that g, g^{jk}_i only depend on x^h_l, and from (3), (4) we have

$$0 = x^h_j\left(\frac{\partial g^{bc}_a}{\partial x^i_j}x^a_{b,c} + \frac{\partial g}{\partial x^i_j}\right) + x^h_{j,k}g^{jk}_i - x^k_{j,i}g^{jh}_k \tag{21}$$

$$0 = x^h_j g^{jk}_i + x^k_j g^{jh}_i. \tag{22}$$

Furthermore, equation (21) is equivalent to the following two equations:

$$0 = x^h_j\frac{\partial g^{bc}_a}{\partial x^i_j} + \delta^h_a g^{bc}_i - \delta^c_i g^{bh}_a \tag{23}$$

$$0 = x^h_j\frac{\partial g}{\partial x^i_j}. \tag{24}$$

From (24) it follows that $g = \lambda \in \mathbb{R}$. Let us fix a frame $u_0 \in F_{x_0}(U)$, and let us choose coordinates (x^i) centred on x_0 such that $u_0 = ((\partial/\partial x^1)_{x_0}, \ldots, (\partial/\partial x^m)_{x_0})$. By evaluating equation (22) at u_0, we obtain

$$g^{hk}_i(u_0) + g^{kh}_i(u_0) = 0 \qquad h \leqslant k. \tag{25}$$

Multiplying equation (23) by ${x_h}^k$ and summing over the index h, we have

$$\frac{\partial g_a^{bc}}{\partial x_k^i} = \delta_i^c {x_h}^k g_a^{bh} - {x_a}^k g_i^{bc}. \tag{26}$$

Taking into account that $\partial {x_l}^i/\partial x_b^a = -{x_a}^i {x_l}^b$, differentiating (26) $r-1$ times and proceeding by recurrence on r, we can conclude that $\partial^r g_a^{bc}/\partial x_{k_1}^{i_1}\dots\partial x_{k_r}^{i_r}$ is a sum of $r!(m+1)$ terms of the form $\pm\delta {x_{\beta_1}}^{\alpha_1}\dots {x_{\beta_r}}^{\alpha_r} g_\alpha^{\beta\gamma}$, δ being the Kronecker symbol of some pair of indices. As $|x_j^i(u_0)| = 1$, there exists a compact neighbourhood Q of u_0 such that $|{x_i}^j(u)| \leqslant 2, \forall u \in Q$. Hence

$$\left|\frac{1}{r!}\frac{\partial^r g_a^{bc}}{\partial x_{k_1}^{i_1}\dots\partial x_{k_r}^{i_r}}(u)\right| \leqslant 2^r M \qquad u \in Q;\ M = (m+1)\max_{\substack{u\in Q\\ \alpha,\beta,\gamma}} |g_\alpha^{\beta\gamma}(u)|.$$

Hence g_a^{bc} is of class C^ω. Evaluating $\partial^r g_a^{bc}/\partial x_{k_1}^{i_1}\dots\partial x_{k_r}^{i_r}$ at u_0, we deduce that each $\partial^r g_a^{bc}/\partial x_{k_1}^{i_1}\dots\partial x_{k_r}^{i_r}(u_0)$ is a linear combination of the $m^2(m-1)/2$ initial values $g_i^{hk}(u_0)$ (see (25)). As g_a^{bc} is analytic, we have $g_i^{jk} = g_a^{bc}(u_0)\varphi_{bc,i}^{a,jk}$ or some functions $\varphi_{bc,i}^{a,jk}$. Hence $\mathcal{L} = g_a^{bc}(u_0)\psi_{bc}^a + \lambda$, with $\psi_{bc}^a = \varphi_{bc,i}^{a,jk} x_{j,k}^i$, and we conclude that the space of invariant Lagrangians with projectable Poincaré–Cartan form is a vector space of dimension $\leqslant 1 + m^2(m-1)/2$. As the Lagrangians $\mathcal{L}_{jk}^i$ are functionally independent (see section 2.2), the dimension must be $1 + m^2(m-1)/2$ exactly, thus finishing the proof. □

3.2. Symmetries and Noether invariants

Let Ω_m be a Lagrangian density on an arbitrary fibred manifold $p\colon P \to M$. A p-projectable vector field $Y \in \mathfrak{X}(P)$ is said to be an infinitesimal symmetry of Ω_m if $L_{Y^{(1)}}\Omega_m = 0$ where $Y^{(1)}$ is the infinitesimal contact transformation attached to Y (e.g., see [6, 8, 24, 28]). Let us denote by $\mathrm{sym}(\Omega_m)$ (or $\mathrm{sym}^v(\Omega_m)$) the Lie algebra of symmetries (or p-vertical symmetries) of Ω_m. We have a semidirect product $\mathrm{sym}(\Omega_{jk}^i) = \mathfrak{X}(M) \times \mathrm{sym}^v(\Omega_{jk}^i)$, with $Y \mapsto (X, Z = Y - \tilde{X})$, X being the projection of Y onto M, given by

$$[(X, Z), (X', Z')] = ([X, X'], [Z, Z'] + [\tilde{X}, Z'] - [\tilde{X}', Z]).$$

Moreover, the Noether theorem holds: if $Y \in \mathrm{sym}(\Omega_m)$, then $\mathrm{d}((j^1 s)^* i_{Y^{(1)}}\Theta) = 0$ for every extremal s [6, 8]. The $(m-1)$-form $i_{Y^{(1)}}\Theta$ is called the Noether invariant associated with the symmetry Y. If Z is another symmetry, we define the Poisson bracket of the two corresponding Noether invariants by the formula $\{i_{Y^{(1)}}\Theta, i_{Z^{(1)}}\Theta\} = i_{[Y,Z]^{(1)}}\Theta$ [6,8,21,22]. Below we determine the symmetries and Noether invariants of Ω_{jk}^i.

Theorem 3.2. *The only π-projectable vector fields on FM that are infinitesimal symmetries of every density Ω_{jk}^i are the natural lifts of vector fields on M; i.e., $\cap_{i,j<k}\mathrm{sym}(\Omega_{jk}^i) = \{\tilde{X} | X \in \mathfrak{X}(M)\}$.*

Proof. As $L_{\tilde{X}^{(1)}}\Omega_{jk}^i = 0$, it suffices to prove that if a π-vertical vector field $Y = v_j^i\partial/\partial x_j^i$, $v_j^i \in C^\infty(FM)$, is a symmetry of all densities Ω_{jk}^i, then $Y = 0$; i.e., $\cap_{i,j<k}\mathrm{sym}^v(\Omega_{jk}^i) = 0$. By imposing the symmetry condition, we obtain $Y^{(1)}(\mathcal{L}_{jk}^i) - \mathcal{L}_{jk}^i {x_l}^h v_h^l = 0$. Substituting $Y^{(1)} = v_b^l\partial/\partial x_b^l + (\partial v_b^l/\partial x^h + x_{e,h}^d\partial v_b^l/\partial x_e^d)\partial/\partial x_{b,h}^l$ and the local expression for $\mathcal{L}_{jk}^i$ (see the formula (6)) into the previous equation, we obtain the following polynomial of first degree in the variables $x_{j,k}^i$ whose coefficients are functions of x^h, x_j^i:

$$\begin{aligned}0 = (v_j^h x_{k,h}^l - v_k^h x_{j,h}^l){x_l}^i - (x_j^h x_{k,h}^l - x_k^h x_{j,h}^l)({x_r}^i {x_l}^s + {x_l}^i {x_r}^s)v_s^r \\ + \left(\frac{\partial v_b^l}{\partial x^h} + x_{e,h}^d\frac{\partial v_b^l}{\partial x_e^d}\right)(x_j^h\delta_k^b - x_k^h\delta_j^b){x_l}^i.\end{aligned}$$

Considering the coefficient of $x^l_{q,h}$, we conclude that this equation is equivalent to the following system:

$$0 = (\delta^q_k v^h_j - \delta^q_j v^h_k){x_l}^i - (\delta^q_k x^h_j - \delta^q_j x^h_k)({x_r}^i {x_l}^s + {x_l}^i {x_r}^s)v^r_s + \left(x^h_j \frac{\partial v^r_k}{\partial x^l_q} - x^h_k \frac{\partial v^r_j}{\partial x^l_q}\right){x_r}^i \tag{27}$$

$$0 = {x_l}^i \left(x^h_j \frac{\partial v^l_k}{\partial x^h} - x^h_k \frac{\partial v^l_j}{\partial x^h}\right). \tag{28}$$

Taking $q \neq j, k$ in (27), we obtain $(x^h_j \partial v^r_k/\partial x^l_q - x^h_k \partial v^r_j/\partial x^l_q){x_r}^i = 0$. Multiplying this equation by x^s_i and by ${x_h}^r$ and summing over the indices i, h, we obtain $\delta^r_j \partial v^s_k/\partial x^l_q = \delta^r_k \partial v^s_j/\partial x^l_q$. As $j \neq k$, we have

$$\frac{\partial v^s_j}{\partial x^l_q} = 0 \qquad \text{for } q \neq j. \tag{29}$$

Now, letting $q = j$ in (27) and taking into account (29), we have

$$-v^h_k {x_l}^i + x^h_k({x_r}^i {x_l}^s + {x_l}^i {x_r}^s)v^r_s - x^h_k \frac{\partial v^r_j}{\partial x^l_j}{x_r}^i = 0.$$

Multiplying the equation above by ${x_h}^k$ and summing over h, k, we have

$$-{x_h}^k v^h_k {x_l}^i + ({x_r}^i {x_l}^s + {x_l}^i {x_r}^s)v^r_s = \frac{\partial v^r_j}{\partial x^l_j}{x_r}^i$$

and multiplying by x^a_i and by x^l_s and summing over i, l, s, we obtain

$$v^a_s = x^l_s \frac{\partial v^a_j}{\partial x^l_j}. \tag{30}$$

Differentiating v^a_s in (30) with respect to x^r_j, $j \neq s$, and taking into account (29), we deduce $0 = \partial v^a_s/\partial x^r_j = x^l_s \partial^2 v^a_j/\partial x^r_j \partial x^l_j$. Hence, from (29) and the equation above, we conclude that $\partial^2 v^a_j/\partial x^l_j \partial x^r_s = 0$. Therefore, the functions $\partial v^a_j/\partial x^l_j$ only depend on x^i. Differentiating (30) with respect to x^r_s, we obtain $\partial v^a_s/\partial x^r_s = \partial v^a_j/\partial x^r_j$. Hence, $v^a_s = g^a_l x^l_s$ for some functions $g^a_l \in C^\infty(M)$. Substituting $g^a_l x^l_s$ for v^a_s in (27) and letting $q = j$, we obtain $0 = -{x_l}^i g^h_t x^t_k + x^h_k({x_r}^i {x_l}^s + {x_l}^i {x_r}^s)g^r_t x^t_s - x^h_k g^r_l {x_r}^i$. Multiplying this equation by x^l_i and summing over i, we have $x^h_k g^r_r - x^t_k g^h_t = 0$. Hence $g^h_t = 0$, thus finishing the proof. □

Proposition 3.3. *Let (E^{i*}_j) be the global basis of $V(FM)$ associated with the standard basis (E^i_j) of $\mathfrak{gl}(m;\mathbb{R})$. A π-vertical vector field of the form $Y = \sum_{i,j} \Phi^i_j E^{i*}_j$, $\Phi^i_j \in C^\infty(M)$, is an infinitesimal symmetry of Ω^1_{23} if and only if the following conditions hold:*

$$\begin{aligned} &\Phi^b_2 = \Phi^b_3 = 0 && b \neq 2, 3 \\ &\Phi^1_a = 0 && a \neq 1 \\ &2\Phi^1_1 + \sum_{r=4}^{m} \Phi^r_r = 0. \end{aligned} \tag{31}$$

Proof. Locally, Y can be written as $Y = x^i_r \Phi^r_j \partial/\partial x^i_j$. Since Y is a symmetry, its components have to satisfy equations (27), (28). From (27) we obtain

${x_a}^1((x^c_2 \Phi^b_3 - x^c_3 \Phi^b_2) + x^c_r(\delta^b_3 \Phi^r_2 - \delta^b_2 \Phi^r_3)) = (\delta^b_3 x^c_2 - \delta^b_2 x^c_3)({x_s}^1 {x_a}^r + {x_a}^1 {x_s}^r)x^s_h \Phi^h_r$.

Proceeding as in the proof of theorem 3.2, we obtain the conditions in the statement. □

Similarly, for the density Ω^1_{12} the following result can be stated:

Proposition 3.4. *A π-vertical vector field of the form $Y = \sum_{i,j} \Phi^i_j E^{i*}_j$, $\Phi^i_j \in C^\infty(M)$, is an infinitesimal symmetry of Ω^1_{12} if and only if the following conditions hold:*

$$\begin{aligned}
&\Phi^b_1 = \Phi^b_2 = 0 \qquad b \neq 1, 2\\
&\Phi^1_a = 0\\
&\sum_{r=3}^m \Phi^r_r = 0.
\end{aligned} \tag{32}$$

We remark that the vector fields Y in proposition 3.3 are the sections of the vector bundle associated with FM and the trivial representation of $GL(m; \mathbb{R})$ on $\mathfrak{gl}(m; \mathbb{R})$. Recall that the vector bundle associated with FM and the adjoint representation of $GL(m; \mathbb{R})$ on $\mathfrak{gl}(m; \mathbb{R})$ is the adjoint bundle; that is, the bundle whose sections are the π-vertical $GL(m; \mathbb{R})$-invariant vector fields on FM, but the only section of the adjoint bundle defining an infinitesimal symmetry of the density Ω^i_{jk} is $Y = 0$.

Proposition 3.5. *If $s = (X_1, \ldots, X_m)$ is an extremal of Ω^1_{23} with dual coframe (ω^i), then $s^*(i_{\tilde{X}}\Theta^1_{23}) = \omega^1 \wedge (i_{[X_3,X]}i_{X_2} - i_{[X_2,X]}i_{X_3})(\omega^1 \wedge \cdots \wedge \omega^m)$ for every $X \in \mathfrak{X}(M)$. Therefore, $s^*(i_{\tilde{X}}\Theta^1_{23}) = 0$ if and only if $\omega^1([X_3, X]) = \omega^1([X_2, X]) = 0$.*

Proof. As in (5), we set $v_{ah} = \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^a} \wedge \cdots \wedge \widehat{\mathrm{d}x^h} \wedge \cdots \wedge \mathrm{d}x^m$, $a < h$. From the formula (20) and the local expression $\tilde{X} = u^i\partial/\partial x^i + x^h_j \partial u^i/\partial x^h \partial/\partial x^i_j$, $u^i \in C^\infty(U)$, we obtain

$$\begin{aligned}
i_{\tilde{X}}\Theta^1_{23} = (-1)^{h-1} \det(x_b{}^a) x_r{}^1 \Bigg(&(x^h_2 x^s_3 - x^h_3 x^s_2) \frac{\partial u^r}{\partial x^s} v_h \\
&+ (x^h_2\, \mathrm{d}x^r_3 - x^h_3\, \mathrm{d}x^r_2) \wedge \left(\sum_{a<h} (-1)^{a-1} u^a v_{ah} + \sum_{a>h} (-1)^a u^a v_{ha} \right) \Bigg).
\end{aligned}$$

Let $(U; x^i)$ be a coordinate system on the domain of the linear frame s so that $X_j = f^i_j \partial/\partial x^i$, $f^i_j \in C^\infty(U)$. We have

$$\begin{aligned}
s^*(i_{\tilde{X}}\Theta^1_{23}) &= (-1)^{h-1} \det(f_b{}^a) \Bigg(f_r{}^1 (f^h_2 f^s_3 - f^h_3 f^s_2) \frac{\partial u^r}{\partial x^s} \\
&\qquad - u^a f_r{}^1 \left(f^h_2 \frac{\partial f^r_3}{\partial x^a} - f^h_3 \frac{\partial f^r_2}{\partial x^a} \right) + u^h \underbrace{f_r{}^1 \left(f^a_2 \frac{\partial f^r_3}{\partial x^a} - f^a_3 \frac{\partial f^r_2}{\partial x^a} \right)}_{\overset{(9)}{=} 0} \Bigg) v_h \\
&= (-1)^{h-1} \det(f_b{}^a) f_r{}^1 \left(f^h_2 \left(f^s_3 \frac{\partial u^r}{\partial x^s} - u^a \frac{\partial f^r_3}{\partial x^a} \right) - f^h_3 \left(f^h_2 \frac{\partial u^r}{\partial x^s} - u^h \frac{\partial f^r_2}{\partial x^h} \right) \right) v_h.
\end{aligned}$$

As $\omega^k = f_h{}^k\, \mathrm{d}x^h$, we have

$$\begin{aligned}
s^*(i_{\tilde{X}}\Theta^1_{23}) &= \omega^1([X_3, X]) i_{X_2}(\omega^1 \wedge \cdots \wedge \omega^m) - \omega^1([X_2, X]) i_{X_3}(\omega^1 \wedge \cdots \wedge \omega^m) \\
&= \omega^1 \wedge (i_{[X_3,X]} i_{X_2} - i_{[X_2,X]} i_{X_3})(\omega^1 \wedge \cdots \wedge \omega^m).
\end{aligned}$$

□

Similarly, for the density Ω^1_{12} the following result can be stated:

Proposition 3.6. *If $s = (X_1, \ldots, X_m)$ is an extremal of Ω^1_{12} with dual coframe (ω^i), then $s^*(i_{\tilde{X}}\Theta^1_{12}) = \omega^1 \wedge (i_{[X_2,X]}i_{X_1} - i_{[X_1,X]}i_{X_2})(\omega^1 \wedge \cdots \wedge \omega^m)$ for every $X \in \mathfrak{X}(M)$. Therefore, $s^*(i_{\tilde{X}}\Theta^1_{12}) = 0$ if and only if $\omega^1([X_2, X]) = \omega^1([X_1, X]) = 0$.*

Theorem 3.7. *The Noether invariant $i_Y\Theta^1_{23}$ (or $i_Y\Theta^1_{12}$) of a π-vertical symmetry of Ω^1_{23} (or Ω^1_{12}) is zero.*

Proof. We only consider the case of the Noether invariant of a π-vertical symmetry of Ω^1_{23}; the other case is dealt with similarly. Proceeding as in the proof of theorem 3.2, we conclude that a π-vertical vector field $Y = v^r_q \partial/\partial x^r_q$, $v^r_q \in C^\infty(FM)$, is a symmetry of Ω^1_{23} if and only if the functions v^r_q satisfy the system (27), (28) for $i = 1$, $j = 2$, $k = 3$. In this case, the equations (27) can be written as follows:

$$A^{hq}_l = \left(x^h_2 \frac{\partial v^r_3}{\partial x^l_q} - x^h_3 \frac{\partial v^r_2}{\partial x^l_q} \right) x_r{}^1 \tag{33}$$

$$\begin{aligned} A^{hq}_l &= -((\delta^q_3 v^h_2 - \delta^q_2 v^h_3) x_l{}^1 - (\delta^q_3 x^h_2 - \delta^q_2 x^h_3) D_l) \\ D_l &= (x_r{}^1 x_l{}^s + x_l{}^1 x_r{}^s) v^r_s. \end{aligned} \tag{34}$$

Let us fix the indices q, l. In matrix notation, equations (33) read

$$\Lambda \cdot \begin{pmatrix} \partial v^1_2/\partial x^l_q \\ \partial v^1_3/\partial x^l_q \\ \vdots \\ \partial v^m_2/\partial x^l_q \\ \partial v^m_3/\partial x^l_q \end{pmatrix} = \begin{pmatrix} A^{1q}_l \\ \vdots \\ A^{mq}_l \end{pmatrix} \tag{35}$$

where Λ is an $m \times 2m$ matrix of rank 2:

$$\Lambda = \begin{pmatrix} -x_1{}^1 x^1_3 & x_1{}^1 x^1_2 & \cdots & -x_m{}^1 x^1_3 & x_m{}^1 x^1_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_1{}^1 x^m_3 & x_1{}^1 x^m_2 & \cdots & -x_m{}^1 x^m_3 & x_m{}^1 x^m_2 \end{pmatrix}.$$

Let $V \subset FM$ be the dense open subset defined by $\Delta = x^1_2 x^2_3 - x^1_3 x^2_2 \neq 0$, $x_1{}^1 \neq 0$. Hence the first two equations in (35) are linearly independent on V. Therefore, the system (35) is compatible if and only if each of its $m - 2$ last equations is a linear combination of the first two equations; that is,

$$A^{hq}_l = \lambda^h A^{1q}_l + \mu^h A^{2q}_l \tag{36}$$

with $\Delta\lambda^h = x^h_2 x^2_3 - x^h_3 x^2_2$, $\Delta\mu^h = x^h_3 x^1_2 - x^h_2 x^1_3$. From (34) we have $A^{hq}_l = 0$ for $q \neq 2, 3$, and equations (36) hold automatically. For $q = 2$, equations (36) read as

$$\begin{aligned} 0 &= \Delta(v^h_3 x_l{}^1 - x^h_3 D_l) - \begin{vmatrix} x^h_2 & x^2_2 \\ x^h_3 & x^2_3 \end{vmatrix} (v^2_3 x_l{}^1 - x^2_3 D_l) - \begin{vmatrix} x^1_2 & x^h_2 \\ x^1_3 & x^h_3 \end{vmatrix} (v^3_3 x_l{}^1 - x^3_3 D_l) \\ &= x_l{}^1 \begin{vmatrix} v^1_3 & v^2_3 & v^h_3 \\ x^1_2 & x^2_2 & x^h_2 \\ x^1_3 & x^2_3 & x^h_3 \end{vmatrix}. \end{aligned}$$

Hence $v^h_3 = \lambda^h v^1_3 + \mu^h v^2_3$ on V. Similarly for $q = 3$, we obtain $v^h_2 = \lambda^h v^1_2 + \mu^h v^2_2$. Taking into account (20) and that $x_h{}^1 \lambda^h = x_h{}^1 \mu^h = 0$, we have

$$\begin{aligned} i_Y \Theta^1_{23} &= (-1)^{i-1} x_j{}^1 \det(x_\beta{}^\alpha)(x^i_2 v^j_3 - x^i_3 v^j_2) v_i \\ &= (-1)^{i-1} x_j{}^1 \det(x_\beta{}^\alpha)(x^i_2(\lambda^j v^1_3 + \mu^j v^2_3) - x^i_3(\lambda^j v^1_2 + \mu^j v^2_2)) v_i = 0. \end{aligned}$$

□

3.3. Jacobi fields

3.3.1. Jacobi equations. Let $\mathcal{S}$ be the sheaf of extremals of a Lagrangian density Ω_m on $p\colon P \to M$; that is, for every open subset $U \subseteq M$, we denote by $\mathcal{S}(U)$ the set of solutions to the Euler–Lagrange equations of Ω_m, which are defined on U. As is well known [6, 8, 28], in the Hamiltonian formalism extremals can be characterized as the solutions to the Hamilton–Cartan equation; that is, s is an extremal if and only if $(j^1 s)^*(i_Y \, d\Theta) = 0$ for all $Y \in \mathfrak{X}(J^1 P)$. The Jacobi fields are the solutions to the linearized Hamilton–Cartan equation. To be precisely, a Jacobi field along an extremal $s \in \mathcal{S}(U)$ is a p-vertical vector field defined along s, $X \in \Gamma(U, s^* V P)$, satisfying the Jacobi equation $(j^1 s)^*(i_Y L_{X^{(1)}} \, d\Theta) = 0$, $\forall Y \in \mathfrak{X}(J^1(p^{-1}U))$. In fact, it is readily checked that if S_t is a one-parameter variation of $s\colon N \to P$ and S_t is an extremal for every t, then the infinitesimal variation X of S_t (i.e., $X \in \Gamma(N, s^* V P)$ is defined by X_x equal to the vector at $t = 0$ tangent to the curve $t \mapsto S_t(x)$, $\forall x \in N$) satisfies the Jacobi equation. Hence we think of the Jacobi fields along s as being the tangent space at s for the 'manifold' $\mathcal{S}(U)$ of extremals and accordingly we denote it by $T_s\mathcal{S}(U)$.

In the particular case of the bundle of linear frames $\pi\colon FM \to M$, a π-vertical vector field X of FM defined along a linear frame $s\colon U \to FM$ is written as $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$, $\Phi^i_j \in C^\infty(U)$.

Theorem 3.8. *Let $s = (X_1, \ldots, X_m)\colon U \to FM$ be an extremal of Ω^1_{23} with dual coframe $(\omega^1, \ldots, \omega^m)$. A π-vertical vector field $X = \sum_{i,j} \Phi^i_j E^{i*}_j|_s$ of FM defined along s is a Jacobi field if and only if it satisfies the following system of $3(m-2)$ linear differential equations:*

$$
\begin{aligned}
0 &= X_a\Phi^1_i - X_i\Phi^1_a + \Phi^h_a\omega^1([X_h, X_i]) - \Phi^1_h\omega^h([X_a, X_i]) - \Phi^1_i\omega^1([X_1, X_a])\\
0 &= 2(X_1\Phi^1_a - X_a\Phi^1_1) + X_l\Phi^l_a - X_a\Phi^l_l + 2(\Phi^h_1\omega^1([X_h, X_a]) - \Phi^h_a\omega^1([X_h, X_1])\\
&\qquad - \Phi^1_h\omega^h([X_1, X_a])) + \Phi^h_l\omega^l([X_h, X_a]) - \Phi^h_a\omega^l([X_h, X_l]) - \Phi^l_h\omega^h([X_l, X_a])\\
0 &= X_2\Phi^j_3 - X_3\Phi^j_2 + \Phi^h_2\omega^j([X_h, X_3])\\
&\qquad - \Phi^h_3\omega^j([X_h, X_2]) - \Phi^j_3\omega^3([X_2, X_3]) - \Phi^j_2\omega^2([X_2, X_3])
\end{aligned}
$$

where $a = 2, 3$, $4 \leqslant i \leqslant m$, $j \neq 2, 3$ and $4 \leqslant l \leqslant m$.

Proof. Let $s = (X_1, \ldots, X_m)$, $X_j = f^i_j \partial/\partial x^i$, be an extremal of Ω^1_{23} defined on an open coordinate subset $U \subseteq M$ and let $X_x = F^i_j(x)\partial/\partial x^i_j|_{s(x)}$, $x \in U$, $F^i_j \in C^\infty(U)$, be a π-vertical vector field on FU defined along s. Taking into account that the Poincaré–Cartan form Θ^1_{23} is defined on FM (see the formula (20)), X is a Jacobi field if and only if $(j^1 s)^*(i_{\partial/\partial x^h_i} L_{X^{(1)}} \, d\Theta^1_{23}) = 0$, for all h, i. By using the local expressions for $X^{(1)}$ and ϑ^k_j, we obtain $L_{X^{(1)}}\vartheta^k_j = 0$, and from (17), (20), we have

$$
d\Theta^1_{23} = \vartheta^j_k \wedge \mathcal{E}^k_j(L^1_{23}) = \vartheta^j_k \wedge (F^{i,kr}_{jq}\vartheta^q_r \wedge v_i + \Psi^k_j \det(x_b{}^a)v). \tag{37}
$$

Hence, as s is an extremal and thus $\Psi^a_b \circ j^1 s = 0$ (see (15), (17)), we obtain

$$
\begin{aligned}
(j^1 s)^*(i_{\partial/\partial x^h_i} L_{X^{(1)}} \, d\Theta^1_{23}) &= (j^1 s)^*(i_{\partial/\partial x^h_i} L_{X^{(1)}}[\Psi^k_j \det(x_b{}^a)\vartheta^j_k \wedge v])\\
&= (j^1 s)^*(i_{\partial/\partial x^h_i}(X^{(1)}(\Psi^k_j \det(x_b{}^a))\vartheta^j_k \wedge v))\\
&= [(X^{(1)}(\Psi^h_i \det(x_b{}^a))) \circ (j^1 s)]v\\
&= [X^{(1)}(\Psi^h_i) \circ (j^1 s)] \det(f_b{}^a)v.
\end{aligned}
$$

Therefore, X is a Jacobi field if and only if $X^{(1)}(\Psi^h_i) \circ (j^1 s) = 0$. From equation (18), we have

$$(X^{(1)}\Psi^i_j)\cdot(x^i_j)+(\Psi^i_j)\cdot(F^i_j)$$

$$=\begin{pmatrix} -2X^{(1)}\mathcal{L}^1_{23} & 0 & 0 & 0 & \dots & 0 \\ X^{(1)}(2\mathcal{L}^1_{31}+\sum_{l=4}^m\mathcal{L}^l_{3l}) & 0 & -X^{(1)}\mathcal{L}^1_{34} & -X^{(1)}\mathcal{L}^1_{35} & \dots & -X^{(1)}\mathcal{L}^1_{3m} \\ -X^{(1)}(2\mathcal{L}^1_{21}+\sum_{l=4}^m\mathcal{L}^l_{2l}) & 0 & X^{(1)}\mathcal{L}^1_{24} & X^{(1)}\mathcal{L}^1_{25} & \dots & X^{(1)}\mathcal{L}^1_{2m} \\ -X^{(1)}\mathcal{L}^4_{23} & 0 & -X^{(1)}\mathcal{L}^1_{23} & 0 & \dots & 0 \\ -X^{(1)}\mathcal{L}^5_{23} & 0 & 0 & -X^{(1)}\mathcal{L}^1_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -X^{(1)}\mathcal{L}^m_{23} & 0 & 0 & 0 & \dots & -X^{(1)}\mathcal{L}^1_{23} \end{pmatrix}.$$

Pulling this equation back along j^1s and recalling that the matrix (f^i_j) is invertible, we conclude that X is a Jacobi field if and only if the following equations hold:

$$\begin{aligned} &X^{(1)}\mathcal{L}^1_{ij}\circ j^1s=0 && 4\leqslant i\leqslant m,\ j=2,3 \\ &(2X^{(1)}\mathcal{L}^1_{j1}+\sum_{l=4}^m X^{(1)}\mathcal{L}^l_{jl})\circ j^1s=0 && j=2,3 \\ &X^{(1)}\mathcal{L}^j_{23}\circ j^1s=0 && j\neq 2,3. \end{aligned} \tag{38}$$

As $E^{k*}_l|_s=f^i_k\partial/\partial x^i_l|_s$, we have $F^i_l=f^i_h\Phi^h_l$. By using the formula (6), we obtain

$$\begin{aligned}(X^{(1)}\mathcal{L}^i_{jk})\circ j^1s &= X_j\Phi^i_k - X_k\Phi^i_j + \Phi^h_j\omega^i([X_h,X_k]) \\ &\quad - \Phi^h_k\omega^i([X_h,X_j]) - \Phi^i_h\omega^h([X_j,X_k])\end{aligned} \tag{39}$$

and the result follows from (38). □

Proceeding similarly for Ω^1_{12}, we obtain

Theorem 3.9. *Let s be an extremal of Ω^1_{12}. A π-vertical vector field X of FM defined along s is a Jacobi field if and only if it satisfies the following equations:*

$$0=X_a\Phi^1_i - X_i\Phi^1_a + \Phi^h_a\omega^1([X_h,X_i]) - \Phi^1_h\omega^h([X_a,X_i])$$

$$0=X_a\Phi^l_l - X_l\Phi^l_a + \Phi^h_a\omega^l([X_h,X_l]) - \Phi^h_l\omega^l([X_h,X_a]) - \Phi^l_h\omega^h([X_a,X_l])$$

$$0=X_1\Phi^j_2 - X_2\Phi^j_1 + \Phi^h_1\omega^j([X_h,X_2]) - \Phi^h_2\omega^j([X_h,X_1]) - \Phi^j_2\omega^2([X_1,X_2])$$

where $a=1,2$, $3\leqslant i\leqslant m$, $j\neq 2$ and $3\leqslant l\leqslant m$.

Note that the only components of the vector field X appearing in the Jacobi equations corresponding to Ω^1_{23} (or Ω^1_{12}) are Φ^1_h; $\Phi^i_a, a=2,3$; $\Phi^i_i, 4\leqslant i\leqslant m$ (or Φ^1_h; $\Phi^i_a, a=1,2$; $\Phi^i_i, 3\leqslant i\leqslant m$) and the rest of components remain completely free.

Corollary 3.10. *A π-vertical infinitesimal symmetry of Ω^1_{23} (or Ω^1_{12}) of the form $Y=\sum_{i,j}\Phi^i_jE^{i*}_j|_s$, $\Phi^i_j\in C^\infty(M)$, is a Jacobi field.*

Proof. Substituting the conditions (31) (or (32)) into the Jacobi equations in theorem 3.8 (or theorem 3.9) and taking into account the equations for the extremals in corollary 2.4 (or corollary 2.6), we have completed our proof. □

In the case of an integrable linear frame $(X_1,\dots,X_m)$, $[X_i,X_j]=0$, the Jacobi equations can be integrated explicitly. In fact, on taking a system of coordinates $(U;x^i)$ such that $X_i=\partial/\partial x^i$, the equations in theorem 3.8 become

$$\frac{\partial\Phi^1_i}{\partial x^a}-\frac{\partial\Phi^1_a}{\partial x^i}=0 \qquad 4\leqslant i\leqslant m,\ a=2,3 \qquad \frac{\partial\Phi^j_3}{\partial x^2}-\frac{\partial\Phi^j_2}{\partial x^3}=0 \qquad j\neq 2,3$$

$$2\left(\frac{\partial\Phi^1_a}{\partial x^1}-\frac{\partial\Phi^1_1}{\partial x^a}\right)=-\sum_{l=4}^m\left(\frac{\partial\Phi^l_a}{\partial x^l}-\frac{\partial\Phi^l_l}{\partial x^a}\right) \qquad a=2,3.$$

From these equations we deduce the existence of functions Φ^j, $j \neq 2, 3$, that parametrize Jacobi fields as follows:

$$\Phi^j_a = \frac{\partial \Phi^j}{\partial x^a} \qquad a = 2, 3\ j \neq 2, 3 \qquad \Phi^1_i = \frac{\partial \Phi^1}{\partial x^i} + \Psi^1_i \qquad 4 \leqslant i \leqslant m \tag{40}$$

$$\Phi^1_1 = \frac{\partial \Phi^1}{\partial x^1} + \frac{1}{2}\sum_{i=4}^{m}\left(\frac{\partial \Phi^i}{\partial x^i} - \Phi^i_i\right) + \Psi \tag{41}$$

where $\Phi^j, \Psi^1_i, \Psi \in C^\infty(U)$ satisfy $\partial\Psi^1_i/\partial x^a = \partial\Psi/\partial x^a = 0$, $a = 2, 3$, and the rest of functions Φ^i_j are arbitrary.

Similarly the Jacobi fields for Ω^1_{12} along an integrable linear frame are parametrized as follows:

$$\Phi^j_a = \frac{\partial \Phi^j}{\partial x^a} \qquad a = 1, 2 \qquad \Phi^1_i = \frac{\partial \Phi^1}{\partial x^i} + \Psi^1_i \qquad 0 = \sum_{l=3}^{m}\left(\frac{\partial \Phi^l}{\partial x^l} - \Phi^l_l\right) + \Psi$$

where $\Phi^j, \Psi^1_i, \Psi \in C^\infty(U)$, $j \neq 2$, $3 \leqslant i \leqslant m$, satisfy $\partial\Psi^1_i/\partial x^a = \partial\Psi/\partial x^a = 0$, $a = 1, 2$, and the rest of the functions Φ^i_j are arbitrary.

3.3.2. Symmetries and Jacobi fields. Let $s\colon M \to P$ be an extremal of a Lagrangian density Ω_m defined on J^1P. The vertical component of a p-projectable vector field $Y \in \mathfrak{X}(P)$ along s is the vector field $Y^v_s \in \Gamma(M, s^*VP)$ (cf section 3.3.1) defined by $(Y^v_s)_x = Y_{s(x)} - s_*(p_*(Y_{s(x)}))$, $\forall x \in M$. As is well known (e.g., see [6, theorem 5.1], [24, theorem 3.11]), the vertical component of an infinitesimal symmetry of Ω_m along an extremal is a Jacobi field. The goal of this section is to study when the converse of this property holds for the densities Ω^1_{23}, Ω^1_{12}. Let us first consider the horizontal symmetries; that is, the symmetries of the form $\tilde{Z}$, where $Z \in \mathfrak{X}(M)$. We have

$$\tilde{Z}^v_s = \left(f^h_j\frac{\partial u^i}{\partial x^h} - u^h\frac{\partial f^i_j}{\partial x^h}\right)\frac{\partial}{\partial x^i_j}\bigg|_s \qquad Z = u^i\frac{\partial}{\partial x^i}$$

with $s = (X_1, \ldots, X_m)$, $X_j = f^i_j\partial/\partial x^i$. In the global basis (E^{r*}_l) (see section 3.3.1), we also have

$$\tilde{Z}^v_s = \sum_{r,l}\Phi^r_l E^{r*}_l|_s \qquad \Phi^r_l = f_i{}^r\left(f^h_l\frac{\partial u^i}{\partial x^h} - \frac{\partial f^i_l}{\partial x^h}u^h\right).$$

Hence

$$\frac{\partial u^s}{\partial x^r} = \left(f^s_h\Phi^h_l + \frac{\partial f^s_l}{\partial x^h}u^h\right)f_r{}^l. \tag{42}$$

This system of m^2 partial differential equations states the necessary conditions for a Jacobi field $Y = \sum_{r,l}\Phi^r_l E^{r*}_l|_s$, $\Phi^r_l \in C^\infty(M)$, to be the vertical component of a horizontal symmetry; i.e., $Y = \tilde{Z}^v_s$.

Theorem 3.11. *Let Y be a π-vertical vector field on FM defined along an extremal $s = (X_1, \ldots, X_m)$ of Ω^1_{23} or Ω^1_{12}. The necessary and sufficient conditions for the system (42) to be completely integrable are the following:*

(1) The extremal $s = (X_1, \ldots, X_m)$ admits a Lie algebra structure (see proposition 2.8): $[X_j, X_k] = c^i_{jk}X_i$.

(2) The intrinsic coefficients Φ^i_j of Y satisfy the following linear system of PDEs:

$$X_q\Phi^p_r - X_r\Phi^p_q + c^p_{qh}\Phi^h_r + c^p_{hr}\Phi^h_q + c^h_{rq}\Phi^p_h = 0 \qquad \forall p, q < r.$$

Then, Y is the vertical component of a horizontal symmetry and, hence, it is a Jacobi field.

Proof. From the very definition, the system (42) is completely integrable if given a point $x \in M$ and arbitrary scalars λ^i, there exists a solution u^i such that $u^i(x) = \lambda^i$. Taking derivatives in (42) with respect to x^q, $q < r$, and imposing the symmetry conditions of the second partial derivatives, after substituting their values deduced from (42) for $\partial u^h/\partial x^q$, $\partial u^h/\partial x^r$, we obtain

$$\left(\frac{\partial f_h^s}{\partial x^q}\Phi_l^h + f_h^s\frac{\partial \Phi_l^h}{\partial x^q} + \frac{\partial^2 f_l^s}{\partial x^h \partial x^q}u^h + \frac{\partial f_l^s}{\partial x^a}\left(f_i^{\,a}\Phi_j^i + \frac{\partial f_j^a}{\partial x^h}u^h\right) f_q^{\,j}\right) f_r^{\,l}$$
$$+ \left(f_h^s\Phi_l^h + \frac{\partial f_l^s}{\partial x^h}u^h\right)\frac{\partial f_r^{\,l}}{\partial x^q}$$
$$= \left(\frac{\partial f_h^s}{\partial x^r}\Phi_l^h + f_h^s\frac{\partial \Phi_l^h}{\partial x^r} + \frac{\partial^2 f_l^s}{\partial x^h \partial x^r}u^h + \frac{\partial f_l^s}{\partial x^a}\left(f_i^{\,a}\Phi_j^i + \frac{\partial f_j^a}{\partial x^h}u^h\right) f_r^{\,j}\right) f_q^{\,l}$$
$$+ \left(f_h^s\Phi_l^h + \frac{\partial f_l^s}{\partial x^h}u^h\right)\frac{\partial f_q^{\,l}}{\partial x^r}.$$

Evaluating at x we obtain an equation for polynomials of degree 1 in the variables $u^i(x)$. Hence the respective coefficients must coincide. As x is any point in M, we obtain the following relations:

$$\left(\frac{\partial^2 f_l^s}{\partial x^h \partial x^q} + \frac{\partial f_l^s}{\partial x^a}\frac{\partial f_j^a}{\partial x^h}f_q^{\,j}\right) f_r^{\,l} + \frac{\partial f_l^s}{\partial x^h}\frac{\partial f_r^{\,l}}{\partial x^q}$$
$$= \left(\frac{\partial^2 f_l^s}{\partial x^h \partial x^r} + \frac{\partial f_l^s}{\partial x^a}\frac{\partial f_j^a}{\partial x^h}f_r^{\,j}\right) f_q^{\,l} + \frac{\partial f_l^s}{\partial x^h}\frac{\partial f_q^{\,l}}{\partial x^r}. \tag{43}$$

$$\left(\frac{\partial f_h^s}{\partial x^q}\Phi_l^h + f_h^s\frac{\partial \Phi_l^h}{\partial x^q} + \frac{\partial f_l^s}{\partial x^a}f_i^{\,a}\Phi_j^i f_q^{\,j}\right) f_r^{\,l} + f_h^s\Phi_l^h\frac{\partial f_r^{\,l}}{\partial x^q}$$
$$= \left(\frac{\partial f_h^s}{\partial x^r}\Phi_l^h + f_h^s\frac{\partial \Phi_l^h}{\partial x^r} + \frac{\partial f_l^s}{\partial x^a}f_i^{\,a}\Phi_j^i f_r^{\,j}\right) f_q^{\,l} + f_h^s\Phi_l^h\frac{\partial f_q^{\,l}}{\partial x^r}. \tag{44}$$

Equation (43) imposes a condition on the linear frame. Let (ω^i) be the dual coframe. Let us fix a point $x \in M$ and let us consider the change of coordinates $\bar{x}^j = a_i^{\,j}x^i$, where $(a_j^{\,i}) = (a_j^i)^{-1}$ and $a_j^i = f_j^i(x)$. Then, $X_j = \bar{f}_j^h\partial/\partial\bar{x}^h$, with $\bar{f}_j^h = a_i^{\,h}f_j^i$, $\bar{f}_j^h(x) = \delta_j^h$. Expressing (43) in the new coordinates, we have

$$(a_q^{\,b}a_r^{\,c} - a_r^{\,b}a_q^{\,c})a_h^{\,d}a_l^s\left(\bar{f}_c^e\frac{\partial^2\bar{f}_e^l}{\partial\bar{x}^b\partial\bar{x}^d} + \frac{\partial\bar{f}_e^l}{\partial\bar{x}^d}\frac{\partial\bar{f}_c^e}{\partial\bar{x}^b} + \bar{f}_b^t\bar{f}_c^e\frac{\partial\bar{f}_t^p}{\partial\bar{x}^d}\frac{\partial\bar{f}_e^l}{\partial\bar{x}^p}\right)(x) = 0.$$

Multiplying this equation, first by $a_a^h a_s^{\,k}$ and summing over h, s, and then by $a_i^q a_j^r$ and summing over b, c, yields

$$\left(\frac{\partial^2\bar{f}_j^k}{\partial\bar{x}^a\partial\bar{x}^i} + \frac{\partial\bar{f}_e^k}{\partial\bar{x}^a}\frac{\partial\bar{f}_j^e}{\partial\bar{x}^i} + \frac{\partial\bar{f}_j^k}{\partial\bar{x}^p}\frac{\partial\bar{f}_i^p}{\partial\bar{x}^a}\right)(x) = \left(\frac{\partial^2\bar{f}_i^k}{\partial\bar{x}^a\partial\bar{x}^j} + \frac{\partial\bar{f}_e^k}{\partial\bar{x}^a}\frac{\partial\bar{f}_i^e}{\partial\bar{x}^j} + \frac{\partial\bar{f}_i^k}{\partial\bar{x}^p}\frac{\partial\bar{f}_j^p}{\partial\bar{x}^a}\right)(x)$$

or equivalently $(X_a(\omega^k([X_i, X_j])))(x) = 0$, $\forall x \in M$. Therefore the functions $\omega^k([X_i, X_j])$ are constant. Hence, if the system (42) is completely integrable, then s admits a Lie algebra structure.

Finally, writing equations (44) in a coordinate system $(U; x^i)$ such that $f_j^i(x) = \delta_j^i$ (i.e., $(X_k)_x = (\partial/\partial x^k)_x$) and evaluating them at x, we obtain

$$\left(\Phi_r^h\frac{\partial f_h^s}{\partial x^q} + \frac{\partial\Phi_r^s}{\partial x^q} + \Phi_q^h\frac{\partial f_r^s}{\partial x^h} + \Phi_h^s\frac{\partial f_r^{\,h}}{\partial x^q} - \Phi_q^h\frac{\partial f_h^s}{\partial x^r} - \frac{\partial\Phi_q^s}{\partial x^r} - \Phi_r^h\frac{\partial f_q^s}{\partial x^h} - \Phi_h^s\frac{\partial f_q^{\,h}}{\partial x^r}\right)(x) = 0$$

i.e.,

$$(X_q\Phi_r^s - X_r\Phi_q^s + \Phi_r^h\omega^s([X_q, X_h]) + \Phi_q^h\omega^s([X_h, X_r]) - \Phi_h^s\omega^h([X_q, X_r]))(x) = 0.$$

□

We remark that the conditions (2) in theorem 3.11 are equivalent to saying that the functions $X^{(1)}\mathcal{L}_{qr}^p$ vanish along $j^1 s$, as follows from equations (39).

Corollary 3.12. *A Jacobi vector field Y along an extremal admitting a Lie algebra structure $s = (X_1, \ldots, X_m)$ of Ω_{23}^1 (or Ω_{12}^1) is the vertical component of a horizontal symmetry if and only if it satisfies the equations in theorem 3.11–(2) except for the following systems of indices: $(p = 1; q = 2, 3; 4 \leqslant r \leqslant m)$, $(p = q = 1; r = 2, 3)$, $(p \neq 2, 3; q = 2; r = 3)$ (or $(p = 1; q = 1, 2; 3 \leqslant r \leqslant m)$, $(p = q = 4; r = 1, 2)$, $(p \neq 1, 2; q = 1; r = 2)$).*

3.4. Pre-symplectic structure

Let Ω_m be a Lagrangian density on an arbitrary fibred manifold $p: P \to M$ and let Θ be the Poincaré–Cartan form associated with Ω_m. Let $X, Y \in T_s\mathcal{S}(U)$ be Jacobi vector fields defined along an extremal $s \in \mathcal{S}(U)$ of Ω_m. Then, $\mathrm{d}[(j^1 s)^*(i_{Y^{(1)}} i_{X^{(1)}}\, \mathrm{d}\Theta)] = 0$ (e.g., see [6]); that is, the $(m-1)$-form $i_{Y^{(1)}} i_{X^{(1)}}\, \mathrm{d}\Theta$ is closed along $j^1 s$. The alternate bilinear map taking values in the closed $(m-1)$-forms:

$$(\omega_2)_s: T_s\mathcal{S}(U) \times T_s\mathcal{S}(U) \longrightarrow Z^{m-1}(U) \qquad (\omega_2)_s(X, Y) = (j^1 s)^*(i_{Y^{(1)}} i_{X^{(1)}}\, \mathrm{d}\Theta)$$

is thus called the pre-symplectic structure associated with Ω_m.

Proposition 3.13. *Let $s = (X_1, \ldots, X_m): U \to FM$ be an extremal of Ω_{23}^1 with dual coframe $(\omega^1, \ldots, \omega^m)$, and let $X = \sum_{i,j} \Phi_j^i E_j^{i*}|_s$, $Y = \sum_{i,j} \Upsilon_j^i E_j^{i*}|_s$ be two Jacobi fields. Then, the pre-symplectic structure associated with Ω_{23}^1 is given by*

$$(\omega_2)_s(X, Y) = \begin{vmatrix} i_{X_2}\omega & i_{X_3}\omega & i_{X_h}\omega \\ \Upsilon_2^1 & \Upsilon_3^1 & \Upsilon_h^1 \\ \Phi_2^h & \Phi_3^h & \Phi_h^h \end{vmatrix} + \begin{vmatrix} i_{X_2}\omega & i_{X_3}\omega & i_{X_h}\omega \\ \Upsilon_2^h & \Upsilon_3^h & \Upsilon_h^h \\ \Phi_2^1 & \Phi_3^1 & \Phi_h^1 \end{vmatrix}$$

where $\omega = \omega^1 \wedge \cdots \wedge \omega^m$. Similarly, the pre-symplectic structure associated with Ω_{12}^1 is given by

$$(\omega_2)_s(X, Y) = \begin{vmatrix} i_{X_1}\omega & i_{X_2}\omega & i_{X_h}\omega \\ \Upsilon_1^1 & \Upsilon_2^1 & \Upsilon_h^1 \\ \Phi_1^h & \Phi_2^h & \Phi_h^h \end{vmatrix} + \begin{vmatrix} i_{X_1}\omega & i_{X_2}\omega & i_{X_h}\omega \\ \Upsilon_1^h & \Upsilon_2^h & \Upsilon_h^h \\ \Phi_1^1 & \Phi_2^1 & \Phi_h^1 \end{vmatrix}.$$

Proof. As Θ_{23}^1 projects onto FM (see (20)), the formula (37) yields

$$i_Y i_X\, \mathrm{d}\Theta_{23}^1 = x_a^j \Phi_k^a i_Y \mathcal{E}_j^k(L_{23}^1) - x_a^j \Upsilon_k^a i_X \mathcal{E}_j^k(L_{23}^1) + \vartheta_k^j \wedge i_Y i_X \mathcal{E}_j^k(L_{23}^1).$$

Pulling this equation back along s, we obtain

$$\begin{aligned} s^*(i_Y i_X\, \mathrm{d}\Theta_{23}^1) &= (-1)^l \det(f_d{}^c)\{\Phi_k^a f_a^j f_i^l(\delta_3^k\Upsilon_2^i - \delta_2^k\Upsilon_3^i) f_j{}^1 - \Upsilon_k^a f_a^j f_i^l(\delta_3^k\Phi_2^i - \delta_2^k\Phi_3^i) f_j{}^1 \\ &\quad - f_a^j f_i^b(\Phi_k^a\Upsilon_h^i - \Upsilon_k^a\Phi_h^i)(\delta_3^k f_2^l - \delta_2^k f_3^l)(f_b{}^1 f_j{}^h + f_j{}^1 f_b{}^h)\} v_l \\ &= (-1)^l \det(f_d{}^c)\left(\begin{vmatrix} f_2^l & f_3^l & f_h^l \\ \Upsilon_2^1 & \Upsilon_3^1 & \Upsilon_h^1 \\ \Phi_2^h & \Phi_3^h & \Phi_h^h \end{vmatrix} + \begin{vmatrix} f_2^l & f_3^l & f_h^l \\ \Upsilon_2^h & \Upsilon_3^h & \Upsilon_h^h \\ \Phi_2^1 & \Phi_3^1 & \Phi_h^1 \end{vmatrix}\right) v_l \end{aligned}$$

and since $i_{X_i}\omega = (-1)^l \det(f_d{}^c) f_i^l v_l$, we have completed our proof. The proof for Ω_{12}^1 is similar. □

We remark that the only components of X, Y appearing in the expression of the presymplectic structure are the same as those appearing in the Jacobi equations in theorems 3.8 and 3.9 (also see the remark following theorem 3.9).

Proposition 3.14. *If a Jacobi field X defined along an extremal s of Ω^1_{23} (or Ω^1_{12}) is an infinitesimal symmetry of this density, then $i_X(\omega_2)_s = 0$. The converse is true if the frame $s = (X_1, \ldots, X_m)$ is integrable.*

Proof. Let $\Theta(\Omega)$ be the Poincaré–Cartan form of a Lagrangian density Ω on a fibred manifold $p\colon P \to M$. We know that for every p-projectable vector field X on P we have $L_{X^{(1)}}\Theta(\Omega) = \Theta(L_{X^{(1)}}\Omega)$. This property is usually referred to as the infinitesimal functoriality of the Poincaré–Cartan form (see [6, 8]). As X is an infinitesimal symmetry, we conclude that $L_X\Theta^1_{23} = 0$ (or $L_X\Theta^1_{12} = 0$). Hence, for every Jacobi field Y, we have $0 = i_Y L_X\Theta^1_{23} = i_Y i_X \,\mathrm{d}\Theta^1_{23} + i_Y \,\mathrm{d}i_X \,\Theta^1_{23}$ (or $0 = i_Y L_X\Theta^1_{12} = i_Y i_X \,\mathrm{d}\Theta^1_{12} + i_Y \,\mathrm{d}i_X \,\Theta^1_{12}$). Moreover, as we proved in theorem 3.7, the Noether invariant of a π-vertical symmetry, vanishes; i.e., $i_X\Theta^1_{23} = 0$ (or $i_X\Theta^1_{12} = 0$), and the first part of the statement follows. As for the second, we give the proof for Ω^1_{23}, the other case being similar. Since $[X_i, X_j] = 0$, once a point $x \in M$ has been fixed, there exists a coordinate system (x^i) centred at x such that $X_i = \partial/\partial x^i$. Using the same notation as above, in this system we have $f^i_j = \delta^i_j$, and from the formula in the proof of proposition 3.13 we conclude that $(\omega_2)_s(X, Y) = 0$ if and only if the following equations hold:

$$\Upsilon^1_2\Phi^1_3 - \Upsilon^1_3\Phi^1_2 = 0$$

$$-2\Upsilon^1_1\Phi^1_a + \Upsilon^1_a\Big(2\Phi^1_1 + \sum_{h=4}^m \Phi^h_h\Big) + \sum_{h=4}^m\big(\Upsilon^h_a\Phi^1_h - \Upsilon^1_h\Phi^h_a - \Upsilon^h_h\Phi^1_a\big) = 0$$

$$\sum_{h=4}^m\big(\Upsilon^1_2\Phi^h_3 - \Upsilon^1_3\Phi^h_2 + \Upsilon^h_2\Phi^1_3 - \Upsilon^h_3\Phi^1_2\big) = 0$$

with $a = 2, 3$. Assume that $(\omega_2)_s(X, Y) = 0$ for every Jacobi field Y. From equations (40), (41), we deduce that the values $\Upsilon^j_a(x)$, $a = 2, 3$, $j \neq 2, 3$, $\Upsilon^1_h(x)$, $\Upsilon^h_h(x)$, $4 \leqslant h \leqslant m$, can be chosen arbitrarily. Hence the Φ^i_j satisfy the equations (31), thus providing our conclusion. □

4. Lower dimensions

For $\dim M = 3, 4$, the equations in corollaries 2.4, 2.6 can be integrated explicitly yielding 'normal forms' for the extremals. As diff M acts on the set of extremals of an invariant Lagrangian (cf proposition 2.1), the general solution to the field equations is then obtained, transforming these normal forms by an arbitrary diffeomorphism. Noether invariants defined by horizontal symmetries are also calculated. Below we summarize these results.

4.1. $\dim M = 3$

Let $s = (X_1, X_2, X_3)$ be an extremal of Ω^1_{23}. Once a point has been fixed in the domain of s, there exists an open coordinate subset $(U; x^i)$ such that

$$X_1 = f\frac{\partial}{\partial x^1} + g\frac{\partial}{\partial x^2} + h\frac{\partial}{\partial x^3} \qquad X_j = g^i_j\frac{\partial}{\partial x^i} \qquad i, j = 2, 3$$

where $f \in C^\infty(\mathbb{R})$, $g, h, g^i_j \in C^\infty(U)$, with $f\det(g^i_j) \neq 0$. Moreover, the Noether invariant associated with $X = u^i\partial/\partial x^i \in \mathfrak{X}(U)$ is $s^*(i_{\tilde X}\Theta^1_{23}) = -f^{-2}\,\mathrm{d}x^1 \wedge \mathrm{d}u^1$.

Similarly, if $s = (X_1, X_2, X_3)$ is an extremal of Ω^1_{12}, we have

$$X_1 = \frac{\partial}{\partial x^1} \qquad X_2 = \sigma\frac{\partial}{\partial x^2} \qquad X_3 = \alpha\frac{\partial}{\partial x^1} + \beta\frac{\partial}{\partial x^2} + \gamma\frac{\partial}{\partial x^3}$$

where $\alpha, \gamma \in C^\infty(\mathbb{R})$, $\sigma, \beta \in C^\infty(U)$, with $\gamma\sigma \neq 0$, and the Noether invariant of X is $s^*(i_{\tilde{X}}\Theta^1_{12}) = \gamma^{-2}(\gamma\, \mathrm{d}u^1 - \alpha\, \mathrm{d}u^3) \wedge \mathrm{d}x^3$.

4.2. $\dim M = 4$

Let $s = (X_1, X_2, X_3, X_4)$ be an extremal of Ω^1_{23}. Once a point has been fixed in the domain of s, there exists an open coordinate subset $(U; x^i)$ such that

$$X_1 = h^i \frac{\partial}{\partial x^i} \qquad X_2 = \frac{\partial}{\partial x^2} \qquad X_3 = f^2 \frac{\partial}{\partial x^2} + f^3 \frac{\partial}{\partial x^3}$$
$$X_4 = g^2 \frac{\partial}{\partial x^2} + g^3 \frac{\partial}{\partial x^3} + g^4 \frac{\partial}{\partial x^4}$$

where $h^i, f^j, g^k \in C^\infty(U)$, $j = 2, 3$, $k > 1$, satisfy $h^1 f^3 g^4 \neq 0$ and $\partial((h^1)^2 g^4)/\partial x^2 = \partial((h^1)^2 g^4)/\partial x^3 = 0$; that is, $(h^1)^2 g^4$ only depends on x^1, x^4. Moreover, the Noether invariant associated with $X = u^i \partial/\partial x^i \in \mathfrak{X}(U)$ is given by $s^*(i_{\tilde{X}}\Theta^1_{23}) = -((h^1)^2 g^4)^{-1}\, \mathrm{d}x^1 \wedge \mathrm{d}u^1 \wedge \mathrm{d}x^4$

Similarly, if $s = (X_1, X_2, X_3, X_4)$ is an extremal of Ω^1_{12}, we have

$$X_1 = \frac{\partial}{\partial x^1} \qquad X_2 = \sigma \frac{\partial}{\partial x^2}$$
$$X_3 = (a^1_1\xi^1 + a^1_2\xi^2)\frac{\partial}{\partial x^1} + f^2 \frac{\partial}{\partial x^2} + \sum_{i=3,4} (a^i_1\xi^1 + a^i_2\xi^2)\frac{\partial}{\partial x^i}$$
$$X_4 = (a^1_1\phi^1 + a^1_2\phi^2)\frac{\partial}{\partial x^1} + g^2 \frac{\partial}{\partial x^2} + \sum_{i=3,4} (a^i_1\phi^1 + a^i_2\phi^2)\frac{\partial}{\partial x^i}$$

where $\sigma, f^2, g^2 \in C^\infty(U)$, and (ξ^1, ϕ^1), (ξ^2, ϕ^2) is the basis of the space of solutions to the system $\partial f/\partial x^2 = \lambda f + \mu g$, $\partial g/\partial x^2 = \alpha f - \lambda g$, determined by the initial conditions $\xi^i(x^1, 0, x^3, x^4) = \delta^i_1$, $\phi^i(x^1, 0, x^3, x^4) = \delta^i_2$, $i = 1, 2$, where the functions $\alpha, \lambda, \mu \in C^\infty(U)$ are arbitrary and a^h_i, $h = 1, 2$, $i = 1, 3, 4$, satisfy

$$\frac{\partial(a^1_1\xi^1 + a^1_2\xi^2)}{\partial x^1} = \frac{\partial(a^1_1\phi^1 + a^1_2\phi^2)}{\partial x^1} = 0$$
$$\frac{\partial a^h_i}{\partial x^2} = 0 \qquad \frac{\partial \delta}{\partial x^1} = 0 \qquad \delta = \begin{vmatrix} a^3_1 & a^3_2 \\ a^4_1 & a^4_2 \end{vmatrix}.$$

In this case, the Noether invariant is

$$s^*(i_{\tilde{X}}\Theta^1_{12}) = \delta^{-2}\left(\begin{vmatrix} a^3_1 & a^3_2 \\ a^4_1 & a^4_2 \end{vmatrix} \mathrm{d}u^1 + \begin{vmatrix} a^4_1 & a^4_2 \\ a^1_1 & a^1_2 \end{vmatrix} \mathrm{d}u^3 + \begin{vmatrix} a^1_1 & a^1_2 \\ a^3_1 & a^3_2 \end{vmatrix} \mathrm{d}u^4 \right) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4.$$

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